ICS 6N Computational Linear Algebra
Vector Space

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Vector Space

Definition:
A vector space is a non empty set V of objects called vectors on which we define two operations: addition and multiplication by a scalar, and they are subject to ten rules:

1) If $u, v \in V$, $u + v \in V$
2) $u + v = v + u$
3) $(u + v) + w = u + (v + w)$
4) There exists a zero vector 0 such that $u + 0 = u$
5) For each $u \in V$, there exists $(-u)$ such that $u + (-u) = 0$
6) For any scalar c and $u \in V$, $cu \in V$
7) $c(u + v) = cu = cv$ for any scalar c
8) $(c + d)u = cu + du$ for any scalars c, d
9) $c(du) = (cd)u$ for any scalars c,d
10) $1u = u$

Vector space examples

- R^n is a vector space.
- P_n: the set of polynomial functions that can be written as

$$f(t) = a_0 + a_1 t + \ldots + a_n t^n$$

then P_n is a vector space if we define:

  - Addition: If $f(t) = a_0 + a_1 t + \ldots + a_n t^n$ and $f \in P_n$, and $g(t) = b_0 + b_1 t + \ldots + b_n t^n$ and $g \in P_n$, then

$$f + g = a_0 + b_0 + (a_1 + b_1)t + \ldots + (a_n + b_n)t^n \in P_n$$

  - Multiplication by a scalar

$$cf(t) = ca_0 + ca_1 t + \ldots + ca_n t^n \in P_n$$

Subspaces

Definition: A subset H of a vector space V is called a **subspace** if:

a) $0 \in H$ and

b) If $x, y \in H$, then $x + y \in H$ and

c) If $x \in H$, then $rx \in H$ for a scalar r

We can also easily see H is also a vector space by itself.

Examples

1) $H = 0$ is a subspace
2) $H = \text{span}\{\text{u}\}$, $u \in R^n$ is a subspace

Null Space of A

- Definition: the null space of $m \times n$ matrix A is the set of all solutions to $Ax = 0$.

$$Null(A) = \{x \in R^n : Ax = 0\}$$

- Null(A) is a subspace of R^n
  - It contains the zero vector.
  - If $x, y \in Null(A)$, does $x + y \in Null(A)$?
  - If $x \in Null(A)$, is $\implies rx \in Null(A)$?

Column Space of A

- Definition: the column space of $m \times n$ matrix A is all the linear combinations of column vectors of A.

$$Col(A) = span\{a_1, \ldots, a_n\}$$

- Col(A) is a subspace of R^m
  - It contains the zero vector.
  - If $x, y \in Col(A)$, does $x + y \in Col(A)$?
  - If $x \in Col(A)$, is $\implies rx \in Col(A)$?

How to describe a subspace?

- Span by a set $x_1, \ldots, x_p \in V$
  $span\left\{x_1, \ldots, x_p\right\}$ = all linear combinations of $x_1, \ldots, x_p$
- $span\left\{x_1, \ldots, x_p\right\}$ is a subspace
- The **spanning set** of H is the set of vectors $x_1, \ldots, x_p$ so that

$$span\{x_1, \ldots, x_p\} = H$$

Example

How to find a spanning set of Null(A)

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & 7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

- reduce it to echelon form

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Find solutions in parametric form

$$x_3 = -2x_4 + 2x_5$$
$$x1 = 2x_2 + x_4 - 3x_5$$

with x_4 and x_5 free.

- Represent the solution in vector form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 u + x_4 v + x_5 w$$

- We have the spanning set

$$Null(A) = span\{u, v, w\}$$

Spanning set representation of null space

- Then number of vectors in the spanning set of Null(A) = number of free variables = n - number of pivot columns
- u, v, w are linearly independent.
  The only way of making $x_2 u + x_4 v + x_5 w = 0$ is if $x_2 = x_4 = x_5 = 0$
- Null(A) = $\{0\}$ if there is no free variables.

Column space

The column space of $m \times n$ matrix A

- A subspace of R^m
- Col A $=$ span$\{a_1, \cdots, a_n\}$
- $Col(A) = R^m \iff Ax = b$ has a solution for every b

Basis of a vector space

Definition: $v_1, v_2, \ldots v_r$ is a **basis** of vector space V if:

a) $V = span\{v_1, \ldots, v_r\}$

b) $\{v_1, v_2, \ldots v_r\}$ is linearly independent

Linear independent

- Definition: $v_1, v_2, \ldots v_r$ are linearly independent if
  $c_1 v_1 + \ldots + c_r v_r = 0$
  has only trivial solutions.
- In other words, v_i cannot be written down as a linear combination of the rest of the preceding vectors for any i. This is
  $v_i \neq c_1 v_1 + \ldots + c_{i-1} v_{i-1}$ for any $i = 1, \ldots, r$

Example

Consider vectors in R^2

$$b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, b_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

- $b_2 = 2b_1 + 2b_2$
- $\{b_1, b_2\}$ is a basis for R^2
- The augmented matrix of $x_1 b_1 + x_2 b_2 + x_3 b_3 = 0$ is

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

  has nontrivial solutions since x_3 is a free variable.
- The first two columns are pivot columns.
- Col A is the span of pivot columns.

Finding a basis of a column space

- Reduce matrix A to echelon form

$$B = \begin{bmatrix} \boxed{1} & 4 & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Find solutions of $Ax = 0$ (also $Bx = 0$) in parametric form

$$x_5 = 0$$
$$x_3 = x_4$$
$$x_1 = -4x_2 - 2x_4$$

  with x_2, x_4 free.
  - With $x_2 = 1, x_4 = 0$: $x_5 = 0, x_3 = 0, x_1 = -4 \implies 4a_1 + a_2 = 0$
  - With $x_2 = 0, x_4 = 1$: $x_5 = 0, x_3 = 1, x_1 = -2 \Rightarrow -2a_1 + a_3 + a_4 = 0$
- Any nonpivot column can be written as a linear combination of pivot columns.

Find basis of column space

- Reduce matrix to echelon form:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \Rightarrow B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Any nonpivot column can be written as a linear combination of pivot columns
- The pivot columns are linearly independent
- The pivot columns form a basis of the column space.
  - Col(A) $= span\{a_1, a_2, a_3, a_4, a_5\}$
  - basis of Col A $= \{a_1, a_3, a_5\}$
  - basis of Col B $= \{b_1, b_3, b_5\}$

- Basis for Null(A):
  Number of vectors in the basis of Null(A) is equal to the number of free variables
- Basis for Col(A):
  Number of vectors in the basis of Col(A) is equal to the number of pivot columns which is equal to the number of basic variables

Dimension

- The dimension of V is the number of vectors in a basis of V
- dim(Null(A)) = number of free variables = number of non-pivot columns
- dim(Col(A)) = number of basic variables
- The rank of A is: r(A) = dim(col(A))

Rank theorem

For any mxn matrix A, $r(A) + \dim(\text{Null}(A)) = n$

- Let A be an nxn matrix. If A is invertible, what is r(A)?
- r(A) $=$ n. And it is called a full rank matrix in this case.
- If a matrix is invertible, the null space contains only the trivial solution and by definition:
  $\text{Null}(A) = \{0\}$
  $\dim(\text{Null}(A)) = 0$

What is the basis for R^n?

One basis (the canonical basis) would be:

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\0\\\vdots\\1\end{bmatrix}$$

So

$x_1 b_1 + x_2 b_2 + \ldots + x_n b_n$

P_n : Polynomial functions up to order n
$f(t) = a_0 + a_1 t + \ldots + a_n t^n$
has a basis
$\{1, t, t^2, \ldots, t^n\}$
$\dim(P_n) =$ n+1

Then any $v \in V$

$v = x_1 b_1 + x_2 b_2 + \ldots + x_r b_r$

coordinates of v in terms of the basis

$$\begin{bmatrix} v \end{bmatrix}_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$B = \begin{bmatrix} b_1 & b_2 & \ldots & b_r \end{bmatrix}$$