ICS 6N Computational Linear Algebra
Vector Equations

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Vectors in R^2

- An example of a vector with two entries is

$$W = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

  where w_1 and w_2 are any real numbers.
- A matrix with only one column is called a column vector, or simply a vector.
- The set of all vectors with 2 entries is denoted by R^2 (read r-two).
- Two vectors are equal if and only if their corresponding entries are equal.
- Given two vectors u and v in R^2, their sum is the vector $u + v$ obtained by adding corresponding entries of u and v.
- Given a vector u and a real number c, the scalar multiple of u by c is the vector cu obtained by multiplying each entry in u by c.

Vector equations

Given $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find 4u, (-3)v and 4u-3v.

Geometric description of R^2

- Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point (a, b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$
- So we may regard R^2 as the set of all points in the plane.

Vectors in R^n

- Let u and v be vectors in R^n

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \in R^n, v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in R^n$$

- $au + cv$ is also a vector in R^n

$$a \times \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + b \times \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a \times u_1 + b \times v_1 \\ a \times u_2 + b \times v_2 \\ \vdots \\ a \times u_n + b \times v_n \end{bmatrix}$$

Algebraic properties of R^n

- The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$.
- For all u, v, w in R^n and all scalars c, d:
  - $u + v = v + u$
  - $(u + v) + w = u + (v + w)$
  - $u + 0 = 0 + u = u$
  - $u + (-u) = -u + u = 0$
  - $c(u + v) = cu + cv$
  - (c+d)u=cu+du
  - c(du) = (cd)u
  - 1u = u

Linear combinations

- Given $v_1, v_2, ..., v_p$ vectors in R^n, and given scalars $c_1, c_2, ..., c_p$, then vector y defined by

$$y = c_1 v_1 + c_2 v_2 + ... + c_p v_p$$

  is called a linear combination of $v_1, v_2, ... v_p$ with weights $c_1, c_2, ..., c_p$
- The weights in a linear combination can be any real numbers, including zero.

Examples

Let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $1v_1 + 2v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a linear combination of v_1 and v_2
- $0v_1 + 0v_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a linear combination of v_1 and v_2

Linear combinations

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether b can be generated (or written) as a linear combination of a_1 and a_2. That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$$

Solution

Given $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$, Can b be written as a linear combination of a_1 and a_2 with weights x_1 and x_2, i.e., $x_1 a_1 + x_2 a_2 = b$?

Solution:

- Write down the augmented matrix of the corresponding linear system. Row reduce it to an echelon form:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} \Longrightarrow \begin{bmatrix} \boxed{1} & 2 & 7 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

- There is a solution since there is no pivot in the last column. (The system is consistent)
- The solution is unique: $x_1 = 3$, $x_2 = 2$. So $b = 3a_1 + 2a_2$

Linear combinations

- A vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

- In particular, b can be generated by a linear combination of $a_1, \cdots, a_n$ if and only if there exists a solution to the linear system corresponding to the above matrix.

Span

- **Definition** If $v_1, v_2, ..., v_p$ are vectors in R^n, then the set of all linear combinations of $v_1, v_2, ..., v_p$, denoted by

$$\text{Span}\{v_1, v_2, ..., v_p\},$$

  is called the subset of R^n spanned (generated) by $v_1, \cdots, v_p$.
- Span $\{v_1, v_2, ..., v_p\}$ is the collection of all vectors that can be written in the form

$$c_1 v_1 + c_2 v_2 + ... + c_p v_p$$

  with $c_1, \cdots, c_p$ scalars.
- The zero vector $\mathbf{0}$ is always in the Span.

Example

If $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is Span$\{v\}$?

Example

- If $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, what is Span$\{v\}$?
- Solution:
  - The collection of all vectors in the form of $cv = \begin{bmatrix} c \\ c \end{bmatrix}$, with any scalar c.
  - Geometrically, it is represented by the line through points $(1, 1)$ and the origin in a plane.

Example

If $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what is Span$\{v_1, v_2\}$?

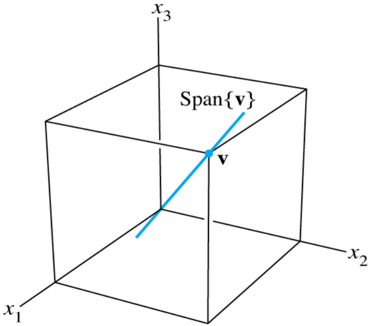
Example

If $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, what is Span$\{v_1, v_2\}$?

Solution: The entire R^2 space.

A GEOMETRIC DESCRIPTION OF SPAN V

- Let v be a nonzero vector in R^3 . Then Span v is the set of all scalar multiples of v, which is the set of points on the line in R^3 through v and 0. See the figure below



A GEOMETRIC DESCRIPTION OF SPAN U, V

- If u and v are nonzero vectors in R^3 , with v not a multiple of u, then Span u, v is the plane in R^3 that contains u, v, and 0.
- In particular, Span u, v contains the line in R^3 through u and 0 and the line through v and 0. See the figure below.