ICS 6N Computational Linear Algebra
The Inverse of a Matrix

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Inverse of a matrix

Definition: Let A be an nxn matrix. A is **invertible** if there exists an nxn matrix C such that

$$CA = AC = I_n$$

If A is invertible, we denote C by A^{-1}, and called it the **inverse** of A.

Example

For example, the following matrix

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

is invertible since we can find a matrix

$$C = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

such that

$$AC = CA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So

$$A^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

The inverse of 2x2 matrices

A 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad - bc \neq 0$, and in this case its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $det(A) = ad - bc$ is called the determinant of A.

The inverse of 2x2 matrices

We can check this

$$AA^{-1} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = I$$

We also obtain the same result for $A^{-1}A$

Solution of Ax=b

- **Theorem** If A is an invertible nxn matrix, then for each b in R^n , the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

- Thus an invertible matrix is row equivalent to an identity matrix.

Solution of Ax=b

- **Theorem** If A is an invertible nxn matrix, then for each b in R^n , the equation $Ax = b$ has the unique solution $x = A^{-1}b$.
- Proof
  - A solution exists because $A(A^{-1}b) = AA^{-1}b = Ib = b$.
  - Uniqueness: If there exists another u with $Au = b$, then $A^{-1}Au = A^{-1}b$. So $u = x$.

Some Properties

- If A is invertible, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

- If A is invertible, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

- If A and B are $n \times n$ invertible matrices, so is AB and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Theorem

Let A be an $n \times n$ matrix, then the following statements are equivalent

a) A is invertible
b) A is row equivalent to an identity matrix
c) A has n pivot columns
d) $Ax = 0$ only has a trivial solution
e) The columns of A are linearly independent
f) A^T is invertible
g) There is an nxn matrix C such that $AC = I$
h) There is an nxn matrix C such that $CA = I$

Elementary Matrices

- An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.
- Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}, A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Compute E_1A, E_2A, E_3A.

Elementary Matrices

- Replacement

$$E_1 A = \begin{bmatrix} a & b & c \\ d & e & f \\ -4a+g & -4b+h & -4ci \end{bmatrix}$$

- Interchange

$$E_2 A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$$

- Scaling

$$E_3 A = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

Elementary Matrices

Elementary matrices are invertible

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix},$$

$$E_2^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

ELEMENTARY MATRICES

- If an elementary row operation is performed on an mxn matrix A, the resulting matrix can be written as EA, where the matrix E is created by performing the same row operation on I_m.
- Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

ELEMENTARY MATRICES

- Theorem: An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Elementary Matrices

If A is invertible, then through a sequence of elementary row operations, we can reduce A to an identity matrix

$$E_p \dots E_2 E_1 A = I$$

Thus

$$A^{-1} = E_p \dots E_2 E_1 = E_p \dots E_2 E_1 I_n$$

can be interpreted as applying the same sequence row operations to I_n.

$$A = E_1^{-1} E_2^{-1} \dots E_p^{-1}$$

General method for obtaining the inverse of a matrix

- Create the augmented matrix with the identity matrix on the right side
- Reduce the matrix to RREF. If

$$\begin{bmatrix} A & I \end{bmatrix} \Longleftrightarrow \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

  Then A is invertible.

Example

Find the inverse of the following matrix, if it exists

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

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Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Uniqueness of matrix inverse

Theorem: If A is invertible, then A^{-1} is unique.

Proof:
Suppose we can find two matrices B and C such that $AB = BA = I$ and $AC = CA = I$. Then

$$B = BI = B(AC) = (BA)C = IC = C$$

which implies that $B = C$. So the inverse must be unique. □

Under what conditions is A invertible?

If A is a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

We want to find a C such that AC=I.
Let $C = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$. So we can see $AC = I$ as two systems of linear equations.

$$Ac_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad Ac_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To solve this we create an augmented matrix but for two systems at the same time, so we consider the augmented matrix for two systems jointly

$$\begin{bmatrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{bmatrix}$$

Example

- Consider

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Reduce the joint augmented matrix to echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

- We have reached RREF and we don't have free variables, so our solution is unique. So

$$c_1 = \begin{bmatrix} -2 \\ \frac{3}{2} \end{bmatrix}, c_2 = \begin{bmatrix} 1 \\ \frac{-1}{2} \end{bmatrix}, C = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

- We can confirm that $AC = CA = I$, so $A^{-1} = C$

Find the inverse of a matrix

- For any nxn matrix A,

$$AC = I \iff Ac_1 = e_1, Ac_2 = e_2, \ldots, Ac_n = e_n$$

where e_i is a column vector whose i-th entry is 1 and other entries are 0.

- Need to solve n systems of linear equations with the same coefficient matrix.
- Method: reduce the following augmented matrix to echelon form

$$\begin{bmatrix} a_1 & \cdots & a_n & e_1 & \cdots & e_n \end{bmatrix}$$

- If all of n systems are consistent, then $A^{-1} = C$.

In which case the system has no solution?

Since A is nxn, there at at most n pivot columns

Case 1: The number of pivots = n

- We wouldn't have zeros in the diagonal of the RREF of A, so we would always have exactly one solution. This means it would be invertible

Case 2: The number of pivots $<$ n

- We would have zeros in the diagonal of the RREF of A, so at least one equation would not have a solution. This means it wouldn't be invertible