ICS 6N Computational Linear Algebra
Matrix Algebra

Xiaohui Xie

University of California, Irvine

xhx@uci.edu

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Matrix

Consider an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n} \end{bmatrix}$$

- a_{ij} is the scalar entry in the ith row and jth column, called the (i, j)-entry.
- Each column is a vector in R^m.
- Two matrices are equal if they have the same size and the corresponding entries are equal
- $a_{11}, a_{22}, \dots$ are called the **diagonal entries**
- A is called diagonal if all non-diagonal entries are zero
  - The identity matrix I_n is a square diagonal matrix with diagonal being 1
- The zero matrix is a matrix in which all entries are zero, written as 0.

Matrix operations

Given two $m \times n$ matrices A and B,

- Sum: $A + B$ is an $m \times n$ matrix whose (i, j)-entry is $a_{ij} + b_{ij}$
- Multiplication by a scalar: $rA = Ar$ is an $m \times n$ matrix whose (i, j)-entry is ra_{ij}, where r is a scalar.
- Matrix vector product: $Ax = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$

Examples

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, compute

- $A + B$
- $2A$
- $A + C$

Examples

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

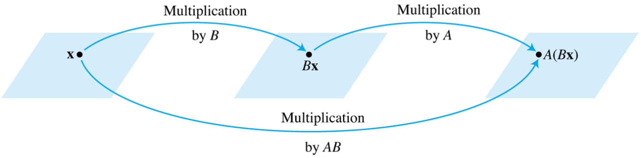
- $A + B = \begin{bmatrix} 2 & 2 & 4 \\ 0 & 6 & 7 \end{bmatrix}$
- $2A = \begin{bmatrix} 2 & 4 & 5 \\ 8 & 10 & 12 \end{bmatrix}$
- $A + C =$ Not defined.

Properties of matrix operations

Given A, B, C matrices of the same size, and scalars r and s,

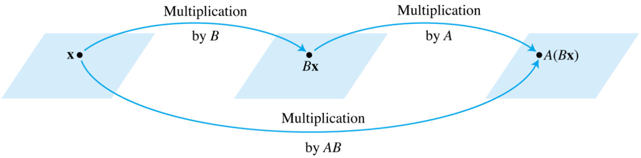
- $(A+B)+C = A+(B+C)$
- $A+B = B+A$
- $A+0 = A$
- $r(A+B) = rA+rB$
- $(r+s)A = rA+sA$
- $r(sA) = (rs)A$

- When a matrix B multiplies a vector x, it transforms x into the vector Bx.
- If this vector is then multiplied in turn by a matrix A, the resulting vector is A(Bx).
- Thus A(Bx) is produced from x by a composition of mappings – the linear transformations.
- Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that $A(Bx) = (AB)x$.



Multiplication by *AB*.

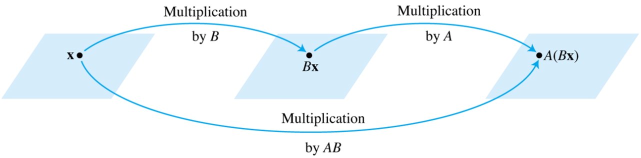
Matrix Multiplication



Multiplication by AB.

- Suppose A is $m \times n$, B is $n \times p$, and x is in R^p
- Denote $B = \begin{bmatrix} b_1 & b_2 & \dots & b_p \end{bmatrix}$.

Matrix Multiplication



Multiplication by AB.

- Suppose A is $m \times n$, B is $n \times p$, and x is in R^p
- Bx is a vector in R^n, A(Bx) is a vector in R^m
- Denote $B = \begin{bmatrix} b_1 & b_2 & \dots & b_p \end{bmatrix}$. Then $Bx = x_1 b_1 + x_2 b_2 + \dots + x_p b_p$

$$\begin{aligned} A(Bx) &= A(x_1 b_1 + x_2 b_2 + \dots + x_p b_p) \\ &= A(x_1 b_1) + A(x_2 b_2) + \dots + A(x_p b_p) \\ &= x_1 (Ab_1) + x_2 (Ab_2) + \dots + x_p (Ab_p) \quad \text{linear combination} \\ &= \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix} x \end{aligned}$$

- So $AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$, an $m \times p$ matrix.

MATRIX MULTIPLICATION

- **Definition**: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product AB is the $m \times p$ matrix whose columns are $Ab_1, \cdots, Ab_p$.
- That is

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

Example

Given $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$, compute AB.

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Solution: $AB = \begin{bmatrix} | & | \\ Ab_1 & Ab_2 \\ | & | \end{bmatrix}$

$Ab_1 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $Ab_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

So $AB = \begin{bmatrix} -1 & 3 \\ 1 & 3 \end{bmatrix}$

Row-column rule for computing AB

- Now let's check the (i, j)-entry of AB:

$$\begin{aligned}(AB)_{ij} &= \text{the } i\text{-th entry of the } j\text{-th column} \\ &= \text{the } i\text{-th entry of } Ab_j \\ &= b_j \cdot \text{ (the } i\text{-th row of } A) \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}\end{aligned}$$

- The (i, j)-entry of AB is the sum of the products of corresponding entries from row i of A and column j of B

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Example

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, compute AB.

Example

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, compute AB.

Solution:

- Are the sizes consistent? Yes
- Based on definition, $AB = \begin{bmatrix} | & | \\ Ab_1 & Ab_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 9 & 2 \end{bmatrix}$
- Or we can also calculate this entry by entry
  $(AB)_{11} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = 3$
  $(AB)_{21} = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = 9$
  And so on until we get
  $AB = \begin{bmatrix} 3 & 2 \\ 9 & 2 \end{bmatrix}$

Special Cases

- An nx1 matrix can be viewed as a vector in R^n (column vector)
- A row vector can be viewed as a 1xn matrix.
- (Dot product) A row vector times a column vector produces a scalar if they are of the same size.

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Special Cases

- (Out product) A column vector times a row vector produces a matrix.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_m b_1 & a_m b_2 & \dots & a_m b_n \end{bmatrix}$$

Special cases

- Let A be an mxn matrix,

$$AI_n = A = I_mA$$

-

$$A0 = 0$$

.

Theorems

If the sizes are consistent

- a) $(AB)C = A(BC)$
- b) $A(B + C) = AB + AC$
- c) $(B + C)A = BA + BC$
- d) $(rA)B = A(rB)$
- e) $I_m A = A I_n$

Warnings

- $AB \neq BA$ in general. They are not even of the same size!

Example

Even if they are the same size it is in general not true

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

If $AB = BA$ then A and B are commutable, but in general they are not.

Warnings

- If $AB = AC$ and $A \neq 0$, we cannot conclude $B = C$

Example
Even if they are the same size it is in general not true
$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$
$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
But we can clearly see $B \neq C$

Powers of a matrix

Definition: If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A.

$$A^k = A \cdots A \quad (k \text{ times})$$

$A^0 = I$ by convention.

Transpose

- Given an mxn matrix A, the transpose of A is the nxm matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.
- If A = $\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, then $A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$
- $(A^T)_{ij} = a_{ji}$

Properties of matrix transpose

If the sizes are consistent

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(rA)B = A(rB)$
- $(AB)^T = B^T A^T$ (note the reverse order!)