ICS 6N Computational Linear Algebra
Systems of Linear Equations

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Linear equations

- Linear algebra was initially developed to solve systems of linear equations

$$
\begin{aligned}
2x - y &= 0 \\
-x + 2y &= 3
\end{aligned}
$$

- We have two equations with two variables x and y, and we want to find a solution
- First we ask, is there a solution?
- If there is the system is called consistent
- Second, if it is consistent, how many solutions?

Algebraic solution: scaling

- The typical process for solving this is to eliminate one of the variables
- First we multiply the second equation by 2 to get an equivalent system of equations (they have the same solution)

$$\begin{aligned} 2x - y &= 0 \\ 2 \times (-x + 2y &= 3) \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} 2x - y &= 0 \\ -2x + 4y &= 6 \end{aligned}$$

Algebraic solution: replacement

- Second we replace the second equation by the addition of the two equations

$$
\begin{aligned}
2x - y &= 0 \\
0x + 3y &= 6
\end{aligned}
$$

- The solution is then y=2, x=1
- There is a solution and it is unique

Geometric interpretation of unique solution

- The solution is the intersection of the lines given by each equation
- We can verify that we got the same solution and it is a unique solution

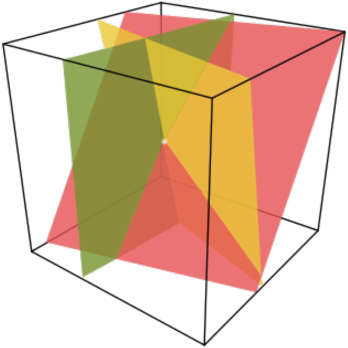
Geometric interpretation of no solution

- Another case is when we have two parallel lines
- In this case there is no solution since the lines never intersect

Geometric interpretation of many solutions

- The third case is when one line overlaps the other
- In this case we have infinitely many solutions

Linear equations with three variables



Linear equations

- A linear equation in the variables $x_1, x_2, \cdots, x_n$ is an equation that can be written in the form

$$a_1 x_1 + a_2 x_2 + \cdots a_n x_n = b$$

  where b and the coefficients $a_1, \cdots, a_n$ are real numbers that are usually known in advance.
- A system of linear equations (or a linear system) is a collection of one or more linear equations involving the same variables say, $x_1, \cdots, x_n$.

Linear equations

- A system of linear equations has
  1. no solution, or
  2. exactly one solution, or
  3. infinitely many solutions.
- A system of linear equations is said to be *consistent* if it has either one solution or infinitely many solutions.
- A system of linear equation is said to be *inconsistent* if it has no solution.

Gaussian Elimination

Eliminate variables through three elementary operations:

- 1) Scale an equation by a nonzero constant: $C \times R1$
- 2) Replace one equation by the sum of itself and a multiple of another equation: replace $R3$ by $R3 + C \times R2$
- 3) Interchange two equations

Gaussian Elimination

Example

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 0 \\
2x_2 - 8x_3 &= 8 \\
-4x_1 + 5x_2 + 9x_3 &= -9
\end{aligned}
$$

Gaussian Elimination

$4 \times eq1 + eq3$ (We eliminate x_1 from equation 3) and we get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned}$$

Gaussian Elimination

$\frac{1}{2} \times eq2$ (we divide equation 2 by 2) and we get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned}$$

$3 \times eq2 + eq3$ and we get

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

From here we know how to solve it using backward substitution

$$\begin{aligned} x_1 &= 2x_2 - x_3 = 29 \\ x_2 &= 4 + 4x_3 = 16 \\ x_3 &= 3 \end{aligned}$$

You can always confirm it is the correct answer by substituting this in the original equation
As we can see this system is consistent and has a unique solution

Gaussian Elimination

Now we are going to use matrices since we only have to keep track of the coefficients in the equation. Consider the previous example.

- Coefficient matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

- Augmented matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Elementary row operations

Elementary row operations include the following:

- (Replacement) Replace one row by the sum of itself and a multiple of another row.
- (Interchange) Interchange two rows.
- (Scaling) Multiply all entries in a row by a nonzero constant.

Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

Elementary row operations

- It is important to note that row operations are reversible.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.
- Two fundamental questions about a linear system are as follows:
  - Is the system consistent; that is, does at least one solution exist?
  - If a solution exists, is it the only one; that is, is the solution unique?

Existence and uniqueness of system of equations

- Example: determine if the following system is consistent

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

- The augmented matrix is

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Gaussian Elimination

- $\xrightarrow{4\times R1+R3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$
- $\xrightarrow{\frac{1}{2}\times R2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$
- $\xrightarrow{3\times R2+R3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
- Notice the diagonal of ones with zeros below (called upper triangular matrix). This is the shape we are looking for.

- The corresponding system of equations would be

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned}$$

- The solution is: $x_3 = 3$, $x_2 = 16$, $x_1 = 29$, which is what we got before

Gaussian Elimination

- Another example

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

- The augmented matrix is $\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$
- We want to generate an upper triangular matrix, so we have to exchange rows

Gaussian Elimination

- $\xrightarrow{exchange R1 and R2} \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$
- $\xrightarrow{\frac{1}{2}\times R1} \begin{bmatrix} 1 & -1.5 & 1 & 0.5 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix}$
- $\xrightarrow{-5\times R1+R3} \begin{bmatrix} 1 & -1.5 & 1 & 0.5 \\ 0 & 1 & -4 & 4 \\ 0 & -0.5 & 2 & -1.5 \end{bmatrix}$
- $\xrightarrow{\frac{1}{2}\times R2+R3} \begin{bmatrix} 1 & -1.5 & 1 & 0.5 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 0 & 2.5 \end{bmatrix}$

We can convert this to equations and we see in the last one that there is no solution for this system of equations

$$\begin{aligned} x_1 - 1.5x_2 + x_3 &= 0.5 \\ x_2 - 4x_3 &= 8 \\ 0 &= 2.5 \end{aligned}$$