ICS 6N Computational Linear Algebra
Eigenvalues and Eigenvectors

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The powers of matrix

- Consider the following dynamic system:

$$x^{(t+1)} = Ax^{(t)}$$

where A is an $n \times n$ matrix and $x^{(t)}$ is vector in R^n.

- How to compute $x^{(100)}$?

$$x^{(t+1)} = A^t x^{(1)}$$

- Need to find ways to efficiently calculate A^t.

Eigenvalues and eigenvectors

- An **eigenvector** of an $n \times n$ matrix A is a nonzero vector x such that

$$Ax = \lambda x$$

  for some scalar λ.
- A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an eigenvector corresponding to λ.

Example

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}, v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$Av = \begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2v$

So $\lambda = 2$ is an eigenvalue, and v is the corresponding eigenvector.

Finding eigenvalues

- $Ax = \lambda x$
  $\iff Ax - \lambda x = 0$
  $\iff Ax - \lambda Ix = 0$
  $\iff (A - \lambda I)x = 0$
- So in order for x to be an eigenvector,
  - x is a nontrivial solution to $(A - \lambda I)x = 0$
  - $\det(A - \lambda I) = 0$

Eigenvalues of triangular matrices

- A is an upper triangular matrix if all values below diagonal are zero; lower triangular if all values above diagonal are zero.
- Determinant of triangular matrix is the product of diagonal entries.
- Eigenvalues of triangular matrix are diagonal entries.

The following statements are equivalent regarding nxn matrix A

1) $Ax = 0$ has nontrivial solutions

2) A is non invertible

3) $\det(A) = 0$

4) $\text{Null}(A) \neq \{0\}$

5) $\dim(\text{Null}(A)) \geq 1$

6) $\text{Rank}(A) < n$

7) The column vectors are linearly dependent

8) $\dim(\text{Col}(A)) < n$

Finding eigenvalues

- Find the roots of the characteristic polynomial equation:

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & \dots & a_{1n} \\ & \ddots & \\ a_{n1} & \dots & a_{nn} - \lambda \end{bmatrix}$$

- Once an eigenvalue is discovered, find its corresponding eigenvector by solving $(A - \lambda I)x = 0$.

Example

$$A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$$

$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix} = -\lambda(3-\lambda) + 2$

$= \lambda^2 - 3\lambda + 2 = 0$

$= (\lambda - 2)(\lambda - 1) = 0$

$\lambda = 1$ or $\lambda = 2$

There is a respective eigenvector for each eigenvalue

For $\lambda = 1$

$$(A - \lambda I)x = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$

$$(A - \lambda I)x = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvector corresponding to eigenvalue λ

- Suppose we have found a λ with $\det(A - \lambda I) = 0$.
- Find its corresponding eigenvector by solving $Ax = \lambda x$
- Any nonzero vector in **Null**$(A - \lambda I)$, called **eigenspace** corresponding to λ, is a corresponding eigenvector

Example

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

- For $\lambda = 2$, the augmented matrix corresponding to $Ax = \lambda x$ is

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Solutions: basic variables: x_1; free variables: x_2 and x_3.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = x_2 v_1 + x_3 v_2$$

- Eigenspace: $\text{span}\{v_1, v_2\}$

Characteristic polynomial

- For matrix A in the previous slide, the characteristic polynomial is of order 3

$$\det(A - \lambda I) = 0 \iff (\lambda - 2)^2(\lambda - c) = 0$$

In this case $\lambda = 2$ has a multiplicity of 2

- In general, the characteristic polynomial of an $n \times n$ matrix A is of order n

$$\det(A - \lambda I) = 0 \iff (\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = 0$$

- The multiplicity of a root is the number of times the root appears in the characteristic polynomial decomposition.

Example

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The characteristic equation: $0 = \begin{vmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}$
- It is an upper triangular matrix. its the determinant is the product of the diagonal entries

$$\det(A - \lambda I) = (\lambda - 5)^2(\lambda - 1)(\lambda - 3)$$

- : Multiplicity of $\lambda = 5$ is 2, while the rest is 1.

Dimension of eigenspace

- If the multiplicity of an eigenvalue is exactly one, then $dim(Null(A - \lambda I)) = 1$, so there is only one eigenvector up to a scale difference.
- Given a eigenvalue λ, $dim(Null(A - \lambda I)) \leq$ its multiplicity

When eigenvalue $\lambda = 0$

The following statements are equivalent:

- $\lambda = 0$
- $\det(A) = 0$
- A is not invertible (also called singular)

Matrix diagonalization

- Suppose nxn matrix A has n linearly independent eigenvectors v_1, v_2, ..., v_n with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (some of these eigenvalues might be equal):

$$Av_1 = \lambda_1 v_1, \cdots, Av_n = \lambda_n v_n$$

- Let $P = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$. Then

$$AP = \begin{bmatrix} Av_1 & Av_2 & \ldots & Av_n \end{bmatrix} = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \ldots & \lambda_n v_n \end{bmatrix} = P\Lambda$$

with $\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$

- $\boxed{A = P\Lambda P^{-1}}$. A is called diagonalizable if this is true.

Calculate the powers of matrix

If A is diagonalizable,

$$\begin{aligned} A^n &= AA\dots A \\ &= P\Lambda P^{-1}P\Lambda P^{-1}\dots P\Lambda P^{-1} \\ &= P\Lambda^n P^{-1} \\ &= P\begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n^n \end{bmatrix} P^{-1} \end{aligned}$$

Solving dynamical systems

- Consider the following discrete dynamical system

$$x_{t+1} = Ax_t$$

with the initial x_0. How to calculate x_t?

- Solution (if A can be diagonalized): $x_t = A^t x_0 = P\Lambda^t P^{-1} x_0$.
- Let $c = P^{-1}x_0$, that is $Pc = x_0$. Then

$$x_t = c_1 \lambda_1^t v_1 + c_2 \lambda_2^t v_2 + \cdots + c_n \lambda_n^t v_n$$

written as a linear combination of eigenvectors.

Suppose A has n linearly independent eigenvectors $v_1, v_2, \ldots, v_n$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$
then A can be diagonalized as:
$A = P\Lambda P^{-1}$ (diagonalization of A)
where

$$P = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

And this is very useful to calculate the power of a matrix

$$A^k = P\Lambda^k P^{-1}$$

$$\Lambda^k = P \begin{bmatrix} (\lambda_1)^n & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\lambda_n)^n \end{bmatrix} P^{-1}$$

Steps for matrix diagonalization

Diagonalize the following matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

- Find eigenvalues $\lambda_1, \lambda_2, \lambda_3$
- Find three linearly independent eigenvectors of A
- Construct $P = [v_1, v_2, v_3]$
- Construct D
- Check $AP = PD$ and $A = PDP^{-1}$

Not all matrices are diagonalizable

An example of non-diagonalizable matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

- Eigenvalues: $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Has only one eigenvalue with multiplicity of 3.
- However,

$$(A - 2I)x = 0 \implies x_2 = 0, x_3 = 0 \text{ with } x_1 \text{ free}$$

- Thus A has only one eigenvector: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and cannot be diagonalized.

Matrix with distinct eigenvalues

- If $v_1, \cdots, v_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \cdots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{v_1, \cdots, v_r\}$ is linearly independent.

- An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Matrices whose eigenvalues are not distinct

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \cdots, \lambda_p$.

- For $1 \leq k \leq p$, the dimension of the eigenspace for $\lambda_k \leq$ the multiplicity of λ_k
- A is diagonalizable if and only if the sum of the dimensions of the eigenvalues equals n.

Application: discrete dynamical systems

- Let's study an ecological problem, in particular, a predator-prey system involving two species: owl and wood rat.
- Denote owl and wood rate populations at time t (unit: month) by

$$x_t = \begin{bmatrix} O_t \\ R_t \end{bmatrix}$$

  where O_t is the number of owls (in unit 1) in the region, and R_t is the number of wood rats (in unit thousands) in the region.
- Suppose the two populations are modeled by

$$O_{t+1} = 0.5 O_t + 0.4 R_t \tag{1}$$

$$R_{t+1} = -0.104 O_t + 1.1 R_t \tag{2}$$

Application: predator-prey system

- Write the population dynamics in matrix format: $x_{t+1} = Ax_t$ with

$$A = \begin{bmatrix} 0.5 & 0.4 \\ -0.104 & 1.1 \end{bmatrix}$$

- A has two eigenvalues $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$ with corresponding eigenvectors

$$v_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}, v_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

- Let $x_0 = c_1 v_1 + c_2 v_2$. Then

$$x_t = c_1 1.02^t v_1 + c_2 0.58^t v_2 \approx c_1 1.02^t v_1$$

  where the approximation is true when t is large.
- In another words, $x_{t+1} = 1.02 x_t$ when t is large.
  - Both species will grow 2% monthly.
  - Every 10 owns, there are about 13 thousands rats.

Application of matrix diagonalization

- Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13 \cdots$$

- How to find the 100-th number x_{100}?

Application of matrix diagonalization

- Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13 \cdots$$

- How to find the 100-th number x_{100} quickly?

Solution

$y_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$

then

$y_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So

$y_{k+1} = \begin{bmatrix} x_{k+1} \\ x_{k+2} \end{bmatrix} = \begin{bmatrix} x_{k+1} \\ x_k + x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$

Then

$y_{k+1} = Ay_k$

So we only have to calculate A to the desired power to solve this. This means we need the diagonalization of A

- From $\det(A - \lambda I) = \lambda^2 - \lambda - 1$, we have

$$\lambda_1 = \frac{1+\sqrt{5}}{2} = 1.618, \lambda_2 = \frac{1-\sqrt{5}}{2} = -0.618$$

- The corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

- Write down y_0 as a linear combination of eigenvectors,

$$y_0 = c_1 v_1 + c_2 v_2$$

that is, solving $Pc = y_0$. So we have $c_1 = 1/\sqrt{5}$ and $c_2 = -c_1$.

- $$\begin{aligned} y_{100} &= c_1 \lambda_1^{100} v_1 + c_2 \lambda_2^{100} v_2 \\ &\approx c_1 \lambda_1^{100} v_1 \end{aligned}$$

- $$x_{100} \approx \frac{1}{\sqrt{5}} \lambda_1^{100} \times 1 = \frac{1.618^{100}}{\sqrt{5}} \approx 3.53 \times 10^{20}$$

Complex eigenvalues

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$\det(A - \lambda I) = \lambda^2 + 1 = 0$$

This means it doesn't have real solutions...
So we have to use complex numbers
Here $\lambda_1 = \sqrt{-1} = i$, and $\lambda_2 = -\sqrt{-1} = -i$
where $i = \sqrt{-1}$ and $i^2 = -1$

Complex numbers

- $z = a + bi$ where a and b are real
  a: represents the real part of z, denoted Re(z)
  b: represents the imaginary part of z, denoted Im(z)
- $z_1 = z_2$ if and only if
  $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$
- Given $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$, definite addition and multiplication by
  - $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
  - $z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$

Complex eigenvalues

The characteristic equation

$$\det(A - \lambda I) = 0$$

is exactly n roots if complex values are allowed.

$$(\lambda - \lambda_1) \dots (\lambda - \lambda_n) = 0$$

Symmetric matrices

- A matrix is called symmetric if $A = A^T$
- The eigenvalues of a symmetric matrix are all real.