ICS 6N Computational Linear Algebra
Row Reduction and Echelon Forms

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Review

- System of linear equations:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{1n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

- We want to answer:
  - Is there a solution?
  - If there is a solution, how many?

The solution of a system of linear equations

Three possible cases:

- a) No solution (Inconsistent)
- b) Exactly one solution (Consistent)
- c) Infinitely many solutions (Consistent)

Find solutions through Gaussian elimination

- Work with augmented matrix A with shape $m \times (n+1)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{1n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

- The goal is to reduce it to an upper triangular form through 3 elementary row operations, maybe something like:

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 & b_1 \\ 0 & 1 & \dots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_m \end{bmatrix}$$

  More precisely, a matrix in row echelon form.

- Finally, use back-substitution to find solutions

Elementary row operations

- Three types:
  - **Scaling**: multiple all elements of a row by a nonzero constant.
  - **Replacement**: Replace one row by the sum of itself and a multiple of another row.
  - **Interchange**: Interchange two rows.
- Two matrices are called *row equivalent* if one can be transformed to another through elementary row operations.
- The corresponding systems of linear equations are also *equivalent*, i.e., having the same solutions.

Echelon Form

- A matrix is in *row echelon form* if
  1. All nonzero rows are above any rows of all zeros, and
  2. Each leading entry of a row (called pivot) is strictly in a column to the right of the leading entry of the row above it.
- These two conditions imply that all entries in a column below a pivot are zeros
- Examples of matrices in row echelon form:

$$\begin{bmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{2} & -8 & 8 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & -3 & 2 & 1 \\ 0 & \boxed{2} & -4 & 8 \\ 0 & 0 & 0 & \boxed{2.5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The marked positions are the *pivot positions*.
- A *pivot column* is a column that contains a pivot.

Reduced Row Echelon Form

- A matrix is in *reduced row echelon form* if it is in row echelon form, and additionally, it satisfies:
  3. The leading entry in each nonzero row is 1.
  4. Each leading 1 is the only nonzero entry in its column
- Examples of matrices in reduced row echelon form:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 8 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduce a matrix to its echelon form

- Gaussian elimination converts a matrix to an equivalent matrix in echelon form:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boxed{1} & 0 & 0 & 29 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

RREF always exists and is unique

- Any nonzero matrix may be row reduced (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations.
- However, each matrix is row equivalent to *one and only one* reduced echelon matrix.

Row reduction algorithm

Reduce a matrix to an echelon form through elementary operations:

1. Begin with the leftmost nonzero column - the first pivot column
2. Select a nonzero entry in the pivot column as a pivot (interchange rows if necessary)
3. Use row replacement to create zeros in positions below the pivot
4. Cover the row containing the pivot position and all rows above it. Repeat steps 1-3 to the remained submatrix.

   $\Longrightarrow$ *row echelon form*

5. Backward phase: Beginning with the rightmost pivot and working upward and to the left,
   - Scale the row containing the pivot to make the leading entry 1
   - Create zeros above the pivot by row replacement

   $\Longrightarrow$ *reduced row echelon form*

Example of row reduction algorithm

- Augmented matrix
$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$
- $\xrightarrow{4\times R1+R3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$
- $\xrightarrow{\frac{1}{2}\times R2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$
- $\xrightarrow{3\times R2+R3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
- This is the row echelon form, now we are going to transform it to reduced row echelon form

Example

- $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
- $\xrightarrow{4\times R3+R2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
- $\xrightarrow{-1\times R3+R1} \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
- $\xrightarrow{2\times R2+R1} \begin{bmatrix} \boxed{1} & 0 & 0 & 29 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$
- This is the reduced row echelon form

More example

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

More example

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & 7 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & 7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- This is now in Reduced Row Echelon Form (RREF).
- Since we only did elementary operations, the solution of the RREF is the same as the solution of the original matrix.
- Columns 1,2,4 are pivot columns since they contain a pivot

$$\begin{bmatrix} \boxed{1} & 0 & -3 & 0 & 5 \\ 0 & \boxed{1} & 2 & 0 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- The basic variables here are: x_1, x_2, x_4
- The free variables are : x_3
- The parametric representation of the solution is:

$$\begin{aligned} x_1 &= 5 + 3x_2 \\ x_2 &= -3 - 2x_3 \\ x_3 &= \mathit{free} \\ x_4 &= 0 \end{aligned}$$

Solutions of linear systems

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$\begin{bmatrix} \boxed{1} & 0 & -5 & 1 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

- The variables x_1 and x_2, corresponding to pivot columns, are called basic variables. The other variable, x_3, is called a free variable.

Solutions of linear systems

Consider the example: $\begin{bmatrix} \boxed{1} & 0 & -5 & 1 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$

- Basic variables: x_1 and x_2, corresponding to pivot columns
- Free variable: x_3
- Key observation: RREF places each basic variable in one and only one equation.
- Solve the reduced system of equations for basic variables in terms of free variables:

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 & \textit{ is free} \end{aligned}$$

- "x_3 is free" means that it can take any value. For example, $x_1 = 1, x_2 = 4, x_3 = 0$ or $x_1 = -4, x_2 = 5, x_3 = -1$

Parametric descriptions of solution sets

- In the previous example, the solution

$$x_1 = 1 + 5x_3$$
$$x_2 = 4 - x_3$$
$$x_3 \textit{ is free}$$

  is a parametric description of the solutions set in which the free variables act as parameters.
- Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Existence and uniqueness theorem

- A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form

$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix}$$

  with b nonzero.
- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Example: unique solution

- Consider the linear system:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

- Row reduction to echelon form

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Longrightarrow \begin{bmatrix} \boxed{1} & 0 & 0 & 29 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

- Basic variables: x_1, x_2, x_3. No free variable.
- Unique solution: $x_1 = 29, x_2 = 16, x_3 = 3$.

Example: infinitely many solutions

- Consider the linear system:

$$\begin{aligned} x_1 + x_2 &= 1 \\ 2x_1 + 2x_2 &= 2 \end{aligned}$$

- Row reduction to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \Longrightarrow \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Basic variable: x_1; free variable: x_2; infinitely many solutions
- Solutions in parametric form: $x_1 = 1 - x_2$, with x_2 free.
- Having a row of zeros reveals redundancy in the system.

Example: no solution

- Consider the linear system

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned}$$

- Row reduction to echelon form:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} \boxed{2} & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & 0 & 0 & \boxed{2.5} \end{bmatrix}$$

- The last column is a pivot column, so the system has no solution $0 = 2.5$ is not true

RREF of linear systems with unique solutions

- If a linear system with m equations and m variables has a unique solution, then its RREF must be

$$\begin{bmatrix} 1 & 0 & \dots & 0 & b_1 \\ 0 & 1 & \ddots & \vdots & b_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & b_m \end{bmatrix}$$

- The coefficient matrix is an identity matrix

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}$$