ICS 6N Computational Linear Algebra
Matrix Determinant

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Matrix Determinant

Assign a scalar to each $n \times n$ matrix A, called **det A**. Require it to satisfy three basic properties:

1. $\det(I_n) = 1$
2. The determinant changes sign when two rows are exchanged.
3. The determinant depends linearly on the first row.

The determinant changes sign when two rows are exchanged

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix}$$

Determinant is linear with respect to any row

- $$\begin{vmatrix} - & r \times a_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_m & - \end{vmatrix} = r \times \begin{vmatrix} - & a_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_m & - \end{vmatrix}$$

$$\begin{vmatrix} - & a_1 + b_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_m & - \end{vmatrix} = \begin{vmatrix} - & a_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_m & - \end{vmatrix} + \begin{vmatrix} - & b_1 & - \\ - & a_2 & - \\ & \vdots & \\ - & a_m & - \end{vmatrix}$$

- For example,

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} 2a & 2b \\ c & d \end{vmatrix} = 2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Warnings

a) $\det(rA) \neq r\det(A)$
   Actually, $\det(rA) = r^n \det(A)$

b) $\det(A + B) \neq \det(A) + \det(B)$

Additional derived properties of determinants

4. If two rows of A are identical, then $\det(A) = 0$

   Proof: if two rows are equal, then interchanging the two rows gives to the same matrix, but changes the sign according to Property 2. Thus $\det(A) = -\det(A)$, so $\det(A) = 0$.

Additional derived properties of determinants

5. Subtracting a multiple of one row from another row (row replacement) leaves the same determinant.

Additional derived properties of determinants

6. If a matrix has a row of zeros, then it's determinant is zero

Additional derived properties of determinants

7. If A is triangular, then $\det(A) = a_{11}a_{22}\dots a_{nn}$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

Proof

I) Assume $a_{11} \neq 0, a_{12} \neq 0, \ldots, a_{nn} \neq 0$

Using row replacement we can reach the RREF of A

$$RREF(A) = \begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & a_{nn} \end{bmatrix}$$

Using property 3

$$\begin{bmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & a_{nn} \end{bmatrix} = a_{11} \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & a_{nn} \end{bmatrix} = \ldots$$

$$= a_{11}a_{22}\ldots a_{nn}det(I) = a_{11}a_{22}\ldots a_{nn}$$

II) Assume at least one of the entries is zero, then we have a row of zeros when we reach RREF(A), so

$$det(A) = a_{11}a_{22}\ldots 0 \ldots a_{nn} = 0$$

Additional derived properties of determinants

8. A is invertible $\iff \det(A) \neq 0$

Check 2x2 matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

For nxn matrices, we use properties of elementary row matrices to prove it.

Determinants of Elementary Matrices

- Interchange

$$E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Scaling

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- Replacement

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Elementary Matrices

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \; E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r \end{bmatrix}$$

Elementary Matrices

$$E_1A = \begin{bmatrix} a & b & c \\ d & e & f \\ -4a+g & -4b+h & -4ci \end{bmatrix} \text{ this is } -4 \times R1 + R3$$

$$E_1I = E_1$$

$$det(I) = det(E_1) = 1$$

$$E_2A = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \text{ this is exchange(R1,R2)}$$

$$E_1I = E_1$$

$$det(E_2) = -1$$

$$E_3A = \begin{bmatrix} a & b & c \\ d & e & f \\ rg & rh & ri \end{bmatrix} \text{ this is scaling } (5 \times R3)$$

Determinants of Elementary Matrices

- For any elementary matrix E,

$$\det E = \begin{cases} 1 & \text{if E is a row replacement} \\ \text{-1} & \text{if E is an interchange} \\ \text{c} & \text{if E is a scaling by c} \end{cases}$$

- Reduce A to its RREF through elementary row operations:

$$E_p \cdots E_1 A = rref(A)$$

$$\det(E_p \cdots E_1 A) = (-1)^r c_1 \cdots c_m \det A$$

where r is the number of row interchanges, and $c_1, \cdots, c_m$ are scaling factors. So

$$\det A = \frac{\det rref(A)}{(-1)^r c_1 \cdots c_m}$$

Proof

Going back...

If A is invertible, $RREF(A) = I_n$, and since we can reach RREF by using the elementary matrices

$E_p \dots E_2 E_1 A = RREF(A)$

$det(E_p \dots E_2 E_1 A) = det(RREF(A))$

$det(E_p) \dots det(E_2) det(E_1) det(A) = det(RREF(A))$

If A is invertible, then $det(RREF(A)) = det(I) = 1$

So $det(A) \neq 0$

If A is not invertible, then the RREF will have a row of zeros so $det(RREF(A)) = det(A) = 0$

Compute det A using row reduction

$$\det A = \begin{cases} (-1)^r c_1^{-1} \cdots c_m^{-1} & \text{if A is invertible} \\ 0 & \text{if A is not invertible} \end{cases}$$

Additional derived properties of determinants

9. $\det(AB) = \det(A)\det(B)$

If A is invertible,

$$\det(I) = \det(A^{-1}A) = \det(A^{-1})det(A) = 1$$

$$\implies \det(A^{-1}) = \frac{1}{\det(A)}$$

Additional derived properties of determinants

10. $\det(A^T) = \det(A)$

Implication: Every rule that applied to rows can be applied to columns!

Cofactor formula: 3x3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11}\begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$a_{11}\begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} + a_{12}\begin{vmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{vmatrix} =$$

$$a_{11}\begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23} \\ 0 & a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \end{vmatrix} =$$

$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Determinants of 3x3 matrices

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{31}a_{12}a_{23} + a_{13}a_{21}a_{32}$$

$$- a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} - a_{33}a_{21}a_{12}$$

Cofactor formula

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) + \ldots + (-1)^{n+1} a_{1n}\det(A_{1n})$$

where A_{ij} denote the submatrix in A after deleting the ith row and jth column.

- Applying to i-th row

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij}\det(A_{ij})$$

- Applying to j-th row

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij}\det(A_{ij})$$