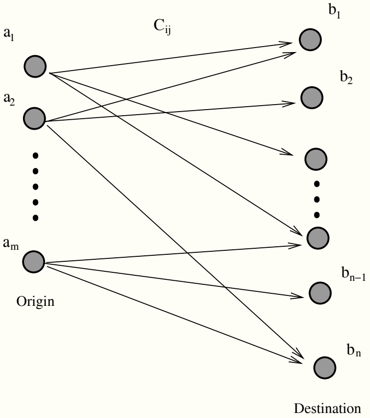
Simplex method for linear programming

slides credit: Weinan E

Outline

Example: Transportation problem

Schematics of transportation problem



Example: Transportation problem

- Formulation:

$$\min s = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraint

$$\sum_{i=1}^{m} x_{ij} \geq b_j, \; j = 1, \ldots, n$$
$$\sum_{j=1}^{n} x_{ij} \leq a_i, \; i = 1, \ldots, m$$
$$x_{ij} \geq 0, \; i = 1, \ldots, m; \; j = 1, \ldots, n.$$

where a_i is the supply of the i-th origin, b_j is the demand of the j-th destinations, x_{ij} is the amount of the shipment from source i to destination j and c_{ij} is the unit transportation cost from i to j.

- Optimization problem (Simplex method)

Linear programming

- Definition:

  If the minimized (or maximized) function and the constraints are all in linear form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n + b.$$

  This type of optimization is called linear programming.

General form of constraints of linear programming

- The minimized function will always be

$$\min_{\boldsymbol{x}} w = \boldsymbol{c}^T \boldsymbol{x} \quad (\text{or } \max)$$

where $\boldsymbol{c}, \boldsymbol{x} \in \mathbb{R}^n$.

- There are 3 kinds of constraints in general:
  - Type I: "$\leq$" type constraint

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

  - Type II: "$=$" type constraint

$$a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = b_j$$

  - Type III: "$\geq$" type constraint

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \geq b_k$$

Examples: general form

- Example 1: (type III)

$$\min w = 100x_1 + 300x_2 + 400x_3 + 75x_4$$
$$s.t. \quad x_1 + 5x_2 + 10x_3 + 0.5x_4 \geq 10000$$
$$x_i \geq 0, \quad i = 1, 2, 3, 4$$

- Example 2: (type I)

$$\max w = 7x + 12y$$
$$s.t. \quad 9x + 4y \leq 360$$
$$4x + 5y \leq 200$$
$$3x + 10y \leq 300$$
$$x \geq 0, y \geq 0$$

- Example 3: (type II)

$$\min w = x_1 + 3x_2 + 4x_3$$
$$s.t. \quad x_1 + 2x_2 + x_3 = 5$$
$$2x_1 + 3x_2 + x_3 = 6$$
$$x_2 \geq 0, x_3 \geq 0$$

Standard form of constraints

- Standard form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots & & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \\ x_i \geq 0, \; i = 1, \ldots, n & & \end{cases}$$

where $b_i \geq 0 \; (i = 1, \ldots, m)$.

- In matrix form

$$\min_{\boldsymbol{x}} w = \boldsymbol{c}^T \boldsymbol{x} \quad (\text{or } \max)$$

Constraints

$$s.t. \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$$

where $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{b} \in \mathbb{R}^m$, $rank(\boldsymbol{A}) = m \leq n$ (This is not essential.) and $b_i \geq 0 \; (i = 1, \ldots, m)$.

Example 1: standard form

- Example 1: (type III)

$$\min w = 100x_1 + 300x_2 + 400x_3 + 75x_4$$

$$s.t.\quad x_1 + 5x_2 + 10x_3 + 0.5x_4 \geq 10000$$

$$x_i \geq 0, \quad i = 1, 2, 3, 4$$

- Introduce surplus variable $x_5 \geq 0$, then the constraint becomes the standard form

$$x_1 + 5x_2 + 10x_3 + 0.5x_4 - x_5 = 10000$$

$$x_i \geq 0, \quad i = 1, 2, 3, 4, 5$$

Example 2: standard form

- Example 2:

$$\max w = 7x + 12y$$
$$s.t.\ \ 9x + 4y \leq 360$$
$$4x + 5y \leq 200$$
$$3x + 10y \leq 300$$
$$x \geq 0, y \geq 0$$

- Introduce slack variable $x_1, x_2, x_3 \geq 0$ and let $x_4 = x, x_5 = y$, then the constraint becomes the standard form

$$\max w = 7x_4 + 12x_5$$
$$\begin{array}{lllll} x_1 & & & +9x_4 + 4x_5 & = & 360 \\ & x_2 & & +4x_4 + 5x_5 & = & 200 \\ & & x_3 & +3x_4 + 10x_5 & = & 300 \end{array}$$
$$x_i \geq 0, i = 1, 2, \ldots, 5$$

Example 3: standard form

- Example 3:

$$\min w = x_1 + 3x_2 + 4x_3$$

$$s.t. \quad x_1 + 2x_2 + x_3 = 5$$

$$2x_1 + 3x_2 + x_3 = 6$$

$$x_2 \geq 0, x_3 \geq 0$$

- Deal with the free variable x_1: Solving x_1 from one equation and substitute it into others.

$$x_1 = 5 - 2x_2 - x_3$$

then

$$\min w = 5 + x_2 + 3x_3$$

$$s.t. \quad x_2 + x_3 = 4$$

$$x_2 \geq 0, x_3 \geq 0$$

Remark

- If some of $b_i < 0$ in the primitive form, we can time -1 to both sides at first and introduce the slack and surplus variables again.

Outline

Definitions

- For the standard form, n is called dimension, m is called order, variables $\boldsymbol{x}$ satisfying constraints

$$s.t. \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0$$

are called feasible solution.

- Suppose $rank(\boldsymbol{A}) = m$, and the first m columns of $\boldsymbol{A}$ are linearly independent, i.e.

$$\boldsymbol{B} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m)$$

is nonsingular, where $\boldsymbol{a}_i = (a_{1i}, a_{2i}, \cdots, a_{mi})$. Then call $\boldsymbol{B}$ a basis.

- The linear system $\boldsymbol{B}\boldsymbol{x}_B = \boldsymbol{b}$ has unique solution $\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$. Define $\boldsymbol{x} = (\boldsymbol{x}_B, \boldsymbol{0})$, then $\boldsymbol{x}$ satisfies

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

$\boldsymbol{x}$ is called a basic solution (the other x_i are 0) with respect to $\boldsymbol{B}$.

Definitions

- If there is 0 among $\boldsymbol{x}_B$, it is called a degenerate basic solution.
- If a basic solution is also a feasible solution, it is called a basic feasible solution.
- x_i corresponding to column indices in $\boldsymbol{B}$ are called basic variable. The others are called non-basic variables.
- The number of the basic feasible solutions is less than

$$C_n^m = \frac{n!}{m!(n-m)!}$$

Example

- Linear programming

$$\max w = 10x_1 + 11x_2$$

$$3x_1 + 4x_2 + x_3 = 9$$

$$5x_1 + 2x_2 + x_4 = 8$$

$$x_1 - 2x_2 + x_5 = 1$$

$$x_i \geq 0, \quad i = 1, 2, 3, 4, 5$$

- Choose $\boldsymbol{B} = (\boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5) = \boldsymbol{I}_{3\times 3}$, then $\boldsymbol{B}$ is a basis,

$$\boldsymbol{x} = (0, 0, 9, 8, 1)$$

is a non-degenerate basic solution. It satisfies the constraint, thus is a basic feasible solution. x_3, x_4, x_5 are basic variables.

Example

- Linear programming

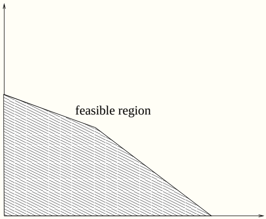
$$\max w = 10x_1 + 11x_2$$

$$3x_1 + 4x_2 \le 17$$

$$2x_1 + 5x_2 \le 16$$

$$x_i \ge 0, \quad i = 1, 2$$

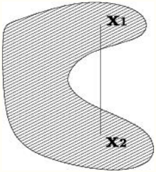
- The set of all the feasible solutions are called feasible region.



- This feasible region is a colorred convex polyhedron spanned by points $\boldsymbol{x}_1 = (0, 0)$, $\boldsymbol{x}_2 = (0, \frac{16}{5})$, $\boldsymbol{x}_3 = (3, 2)$ and $\boldsymbol{x}_4 = (\frac{17}{3}, 0)$.

Definitions

- A convex set S means for any $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in S$ and $\lambda \in [0, 1]$, then $\boldsymbol{x} = \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2 \in S$. A non-convex set is shown below.

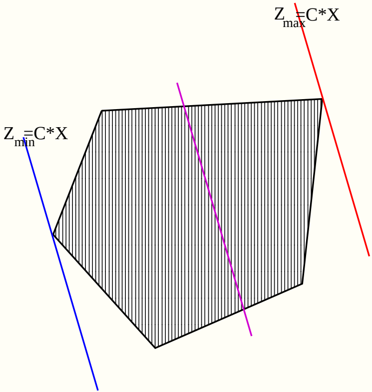


- Graphically, convex means any line segment $\overline{\boldsymbol{x}_1 \boldsymbol{x}_2}$ belongs to S if $\boldsymbol{x}_1$, $\boldsymbol{x}_2 \in S$.
- The vertices $\boldsymbol{x}_1 = (0, 0)$, $\boldsymbol{x}_2 = (0, \frac{16}{5}), \ldots$ are called extreme points because there is no $\boldsymbol{y}_1, \boldsymbol{y}_2 \in S$, $\boldsymbol{y}_1 \neq \boldsymbol{y}_2$ and $0 < \lambda < 1$, such that $\boldsymbol{x}_i = \lambda \boldsymbol{y}_1 + (1 - \lambda)\boldsymbol{y}_2$.

Fundamental theorem

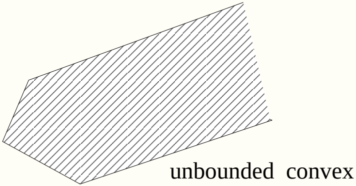
Theorem (Fundamental theorem)

Optimizing a linear objective function $w = \boldsymbol{c}^T \boldsymbol{x}$ is achieved at the extreme points in the feasible region colorblue if the feasible solution set is not empty and the optimum is finite.



Some basic theorems

- There are three cases for the feasible solutions of the standard form
  - Empty set;
  - Unbounded set;



  - Bounded convex polyhedron.
- A point in the feasible solution set is a extreme point if and only if it is a basic feasible solution.

Outline

Simplex method

- Simplex method is first proposed by G.B. Dantzig in 1947.
- Simply searching for all of the basic solution is not applicable because the whole number is C_n^m.
- Basic idea of simplex: Give a rule to transfer from one extreme point to another such that the objective function is decreased. This rule must be easily implemented.

Canonical form

- First suppose the standard form is

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \quad \boldsymbol{x} \geq 0$$

- One canonical form is to transfer a coefficient submatrix into $\boldsymbol{I}_m$ with Gaussian elimination. For example $\boldsymbol{x} = (x_1, x_2, x_3)$ and

$$(\boldsymbol{A}, \boldsymbol{b}) = \begin{pmatrix} 1 & 1 & 1 & 5 \\ 1 & 2 & 0 & 4 \end{pmatrix} \to \boldsymbol{B} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 0 & 4 \end{pmatrix}$$

  then it is a canonical form for x_1 and x_3. One extreme point is $\boldsymbol{x} = (4, 0, 1)$, x_1 and x_3 are basic variables.

Transfer

- Now suppose $\boldsymbol{A}$ is in canonical form as the last example, then we transfer from one basic solution to another.
- Choose $\boldsymbol{a}_2$ to enter the basis and $\boldsymbol{a}_1$ leave the basis.

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 0 & 4 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & 1 & 1 \\ 0.5 & 1 & 0 & 2 \end{pmatrix}$$

$$\to \begin{pmatrix} 0.5 & 0 & 1 & 3 \\ 0.5 & 1 & 0 & 2 \end{pmatrix}$$

- The canonical form for x_2 and x_3. The basic solution is $\boldsymbol{x} = (0, 2, 3)$. It is also a extreme point.

Transfer

- The transferred basic solution may be not feasible in general.
  1. How to make the transferred basic solution feasible?
  2. How to make the objective function decreasing after transfer?

How to make the transferred basic solution feasible?

- Assumption: All of the basic feasible solutions are non-degenerate. i.e. if $\boldsymbol{x} = (x_1, x_2, \ldots, x_m, 0, \ldots, 0)$ is a basic feasible solution, then $x_i > 0$.
- Suppose the basis is $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m\}$ initially, and select $\boldsymbol{a}_k$ $(k > m)$ enter the basis. Suppose

$$\boldsymbol{a}_k = \sum_{i=1}^{m} y_{ik}\boldsymbol{a}_i$$

then for any $\epsilon > 0$

$$\epsilon\boldsymbol{a}_k = \sum_{i=1}^{m} \epsilon y_{ik}\boldsymbol{a}_i$$

- $\boldsymbol{x}$ is a basic feasible solution

$$\sum_{i=1}^{m} x_i\boldsymbol{a}_i = \boldsymbol{b}$$

How to make the transferred basic solution feasible?

- We have

$$\sum_{i=1}^{m}(x_i - \epsilon y_{ik})\boldsymbol{a}_i + \epsilon \boldsymbol{a}_k = \boldsymbol{b}$$

- Because $x_i > 0$, if $\epsilon > 0$ is small enough,

$$\tilde{\boldsymbol{x}} = (x_1 - \epsilon y_{1k}, x_2 - \epsilon y_{2k}, \ldots, x_m - \epsilon y_{mk}, 0, \ldots, 0, \epsilon, 0, \ldots, 0)$$

  is a feasible solution.

- To make it a basic solution we choose

$$\epsilon = \min_{1\le i\le m}\left\{\frac{x_i}{y_{ik}}\middle| y_{ik} > 0\right\} = \frac{x_r}{y_{rk}}$$

  then $\tilde{\boldsymbol{x}}$ is a basic feasible solution, and let $\boldsymbol{a}_r$ leave the basis.

- If $y_{ik} \le 0$ for $i = 1, 2, \ldots, m$, then for any $\epsilon > 0$, $\tilde{\boldsymbol{x}}$ is feasible, thus the feasible region is unbounded in this case.

How to make the transferred basic solution feasible?

- Suppose $n = 6$ and the constraints

$$(\boldsymbol{A}, \boldsymbol{b}) = \begin{pmatrix} 1 & 0 & 0 & 2 & 4 & 6 & 4 \\ 0 & 1 & 0 & 1 & 2 & 3 & 3 \\ 0 & 0 & 1 & -1 & 2 & 1 & 1 \end{pmatrix}$$

- One basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$, the basic feasible solution

$$\boldsymbol{x} = (4, 3, 1, 0, 0, 0)$$

We want to choose $\boldsymbol{a}_4$ enter the basis. The problem is to choose $\boldsymbol{a}_i$ leave the basis.

- Compute $\frac{x_i}{y_{ik}}$ $(k = 4)$

i	1	2	3
$\frac{x_i}{y_{ik}}$	2	3	*

so let $\boldsymbol{a}_1$ enter into basis.

- The new basic solution for x_2, x_3, x_4 is

$$\boldsymbol{x} = (0, 1, 3, 2, 0, 0).$$

How to make the objective function decreasing after transfer?

- The aim is to choose k such that the objective function decreasing after $\boldsymbol{a}_k$ enter the basis.
- Suppose the canonical form is

$$x_i + \sum_{j=m+1}^{n} y_{ij} x_j = y_{i0}, \quad i = 1, 2, \ldots, m$$

where $y_{i0} > 0$. The basic feasible solution

$$\boldsymbol{x} = (y_{10}, y_{20}, \ldots, y_{m0}, 0, \ldots, 0)$$

the value of objective function

$$z_0 = \boldsymbol{c}_B^T \boldsymbol{x}_B = \sum_{k=1}^{m} c_k y_{k0}$$

How to make the objective function decreased after transfer?

- For any feasible solution $\boldsymbol{x} = (x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)$, we have

$$\begin{aligned} z &= \sum_{k=1}^{m} c_k(y_{k0} - \sum_{j=m+1}^{n} y_{kj}x_j) + \sum_{j=m+1}^{n} c_j x_j \\ &= \sum_{k=1}^{m} c_k y_{k0} + \sum_{j=m+1}^{n} c_j x_j - \sum_{j=m+1}^{n} (\sum_{k=1}^{m} c_k y_{kj}) x_j \\ &= z_0 + \sum_{j=m+1}^{n} (c_j - \sum_{k=1}^{m} c_k y_{kj}) x_j \\ &= z_0 + \sum_{j=m+1}^{n} (c_j - z_j) x_j \end{aligned}$$

where $z_j = \boldsymbol{c}_B^T \boldsymbol{y}_j = \sum_{k=1}^{m} c_k y_{kj}$.

- If there exists j $(m+1 \leq j \leq n)$ such that $r_j = c_j - z_j < 0$, then when x_j change from 0 to positive, the objective function will be decreased.

Simplex strategy

- Optimality criterion: If $r_j \geq 0$ for all j, then it is a optimal feasible solution.
- Unbounded criterion: If for some k $(r_k < 0)$, we have $y_{jk} \leq 0$ $(j = 1, 2, \cdots, m)$, then $\min z = -\infty$.
- Otherwise: We can choose the vector $\boldsymbol{a}_k$ $(r_k < 0)$ to enter the basis and the vector $\boldsymbol{a}_j$ $(\frac{y_{j0}}{y_{jk}} = \min_i \frac{y_{i0}}{y_{ik}},\ y_{ik} > 0)$ leave the basis.

Example

- Example

$$\min z = -(3x_1 + x_2 + 3x_3)$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leq \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, \boldsymbol{x} \geq 0$$

- Step 1: change into standard form

$$\min z = -(3x_1 + x_2 + 3x_3)$$

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}, x_i \geq 0, \quad i = 1, 2, \ldots, 6$$

Example

- Step 2: Choose x_4, x_5, x_6 as basic variables, and compute the test number

$$r_1 = c_1 - z_1 = -3, \quad r_2 = c_2 - z_2 = -1, \quad r_3 = c_3 - z_3 = -3.$$

set up simplex tableau

Basis	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
$\boldsymbol{a}_4$	2	1	1	1	0	0	2
$\boldsymbol{a}_5$	1	2	3	0	1	0	5
$\boldsymbol{a}_6$	2	2	1	0	0	1	6
r_j	-3	-1	-3	0	0	0	$z_0 = 0$

Example

- Step 3: Choose vector to enter the basis. Because $r_j < 0,\ j = 1, 2, 3$, any one among $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3$ could enter the basis. We choose $\boldsymbol{a}_2$ (in general, $\boldsymbol{a}_1$ or $\boldsymbol{a}_3$ will be chosen because -3 is smaller).
- Step 4: Choose vector to leave the basis. Compute $\frac{y_{i0}}{y_{ik}},\ y_{ik} > 0$, $k = 2,\ i = 1, 2, 3$, we have

$$\frac{y_{10}}{y_{12}} = 2, \quad \frac{y_{20}}{y_{22}} = 2.5, \quad \frac{y_{30}}{y_{32}} = 3$$

  Thus $\boldsymbol{a}_4$ leave the basis.
- Step 5: Perform Gaussian elimination to obtain a new canonical form for basis $\boldsymbol{a}_2, \boldsymbol{a}_5, \boldsymbol{a}_6$ and set up simplex tableau.

Basis	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
$\boldsymbol{a}_2$	2	1	1	1	0	0	2
$\boldsymbol{a}_5$	-3	0	1	-2	1	0	1
$\boldsymbol{a}_6$	-2	0	-1	-2	0	1	2
r_j	-1	0	-2	1	0	0	$z_0 = -2$

Example

- Step 6: Choose vector to enter the basis. Because $r_j < 0, \ j = 1, 3$, any one among $\boldsymbol{a}_1, \boldsymbol{a}_3$ could enter the basis. We choose $\boldsymbol{a}_3$.
- Step 7: Choose vector to leave the basis. Compute $\frac{y_{i0}}{y_{ik}}, \ y_{ik} > 0$, $k = 3, \ i = 1, 2, 3$, we have $(y_{i3} > 0, \ i = 1, 2)$

$$\frac{y_{10}}{y_{13}} = 2, \quad \frac{y_{20}}{y_{23}} = 1$$

  Thus $\boldsymbol{a}_5$ leave the basis.
- Step 8: Perform Gaussian elimination to obtain a new canonical form for basis $\boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_6$ and set up simplex tableau.

Basis	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
$\boldsymbol{a}_2$	5	1	0	3	-1	0	1
$\boldsymbol{a}_3$	-3	0	1	-2	1	0	1
$\boldsymbol{a}_6$	-5	0	0	-3	2	1	4
r_j	-7	0	0	-3	2	0	$z_0 = -4$

Example

- Step 9: Choose vector to enter the basis. Because $r_j < 0,\ j = 1, 4$, any one among $\boldsymbol{a}_1, \boldsymbol{a}_4$ could enter the basis. We choose $\boldsymbol{a}_1$.
- Step 10: Choose vector to leave the basis. Compute $\frac{y_{i0}}{y_{ik}},\ y_{ik} > 0$, $k = 1,\ i = 1, 2, 3$, we have $(y_{i1} > 0,\ i = 1)$

$$\frac{y_{10}}{y_{11}} = \frac{1}{5}$$

  Thus $\boldsymbol{a}_2$ leave the basis.
- Step 11: Perform Gaussian elimination to obtain a new canonical form for basis $\boldsymbol{a}_1, \boldsymbol{a}_3, \boldsymbol{a}_6$ and set up simplex tableau.

Basis	$\boldsymbol{a}_1$	$\boldsymbol{a}_2$	$\boldsymbol{a}_3$	$\boldsymbol{a}_4$	$\boldsymbol{a}_5$	$\boldsymbol{a}_6$	$\boldsymbol{b}$
$\boldsymbol{a}_1$	1	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$
$\boldsymbol{a}_3$	0	$\frac{3}{5}$	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{8}{5}$
$\boldsymbol{a}_6$	0	1	0	-1	0	1	4
r_j	0	$\frac{7}{5}$	0	$\frac{6}{5}$	$\frac{3}{5}$	0	$z_0 = -\frac{27}{5}$

Example

- Step 9: Choose vector to enter the basis. Because $r_j > 0,\ j = 1, 3, 6$, so we obtain the optimal solution $z^* = -\frac{27}{5}$, and the corresponding extreme point is

$$\boldsymbol{x} = (\frac{1}{5}, 0, \frac{8}{5}, 0, 0, 4)$$

Initial basic feasible solution — two step method

- An auxiliary problem ($\boldsymbol{y} \in \mathbb{R}^m$)

$$\min z = \sum_{i=1}^{m} y_i$$

$$\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}$$

$$\boldsymbol{x} \geq 0, \quad \boldsymbol{y} \geq 0$$

- The initial basic feasible solution is trivial

$$(\boldsymbol{x}, \boldsymbol{y}) = (0, \boldsymbol{b})$$

- Theorem: If the optimal feasible solution of the auxiliary problem is $(\boldsymbol{x}^*, 0)$, then $\boldsymbol{x}^*$ is a basic feasible solution of the primitive problem; if the optimal feasible solution of the auxiliary problem is $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, $\boldsymbol{y}^* \neq 0$, then there is no feasible solution for the primitive problem.

What about the degenerate basic feasible solution?

- In general, the strategy of leaving and entering basis is chosen as
  1. If more than one index j such that $r_j < 0$, let
     $$r_k = \min\{r_j \mid r_j < 0\}$$
     choose $\boldsymbol{a}_k$ to enter the basis;
  2. If
     $$\min\left\{\frac{y_{i0}}{y_{ik}} \,\middle|\, y_{ik} > 0\right\} = \frac{y_{r_1 0}}{y_{r_1 k}} = \cdots = \frac{y_{r_t 0}}{y_{r_t k}}$$
     and $r_1 < \cdots < r_t$, then choose $\boldsymbol{a}_{r_1}$ to leave the basis.
- For degenerate case, cycling will appear for this strategy!

What about the degenerate basic feasible solution?

- Bland's method: Change the strategy of leaving and entering basis into
  1. If more than one index j such that $r_j < 0$, let
  $$k = \min\{j \mid r_j < 0\}$$
  choose $\boldsymbol{a}_k$ to enter the basis;
  2. If
  $$\min\left\{\frac{y_{i0}}{y_{ik}} \,\middle|\, y_{ik} > 0\right\} = \frac{y_{r_1 0}}{y_{r_1 k}} = \cdots = \frac{y_{r_t 0}}{y_{r_t k}}$$
  and $r_1 < \cdots < r_t$, then choose $\boldsymbol{a}_{r_1}$ to leave the basis (the same as before).
- Bland's method could eliminate the cycling, but it needs more computational effort.

Comment on simplex method

- In 1972, V. Klee and G. Minty constructed a linear programming problem which need $O(2^n)$ simplex steps! This shows simplex method is not a polynomial method.
- The first polynomial-time LP algorithm was devised by L. Khachian (USSR) in 1979. His ellipsoid method is $O(n^6)$. Though his method is faster than simplex method theoretically, real implementations show counter results.
- In 1984, N. Karmarkar announced a polynomial-time LP method which is $O(n^{3.5})$. This begins the interior point revolution. Interior point method was faster than simplex for some very large problems, the reverse is true for some problems, and the two approaches are more or less comparable on others.