CS206 Principles of Scientific Computing
Review of Linear Algebra

Xiaohui Xie

University of California, Irvine

xhx@ics.uci.edu

System of linear equations

m equations

n variables

- System of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{1n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

- We want to answer:
  - Is there a solution?
  - If there is a solution, how many?

The solution of a system of linear equations

Three possible cases:

- a) No solution (Inconsistent)
- √ b) Exactly one solution (Consistent)
- c) Infinitely many solutions (Consistent)

Find solutions through Gaussian elimination

- Work with augmented matrix A with shape $m \times (n+1)$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{1n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{bmatrix}$$

- The goal is to reduce it to an upper triangular form through 3 elementary row operations, maybe something like:

$$A = \begin{bmatrix} 1 & 0 & \dots & 0 & b_1 \\ 0 & 1 & \dots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b_m \end{bmatrix}$$

More precisely, a matrix in row echelon form.
- Finally, use back-substitution to find solutions

Elementary row operations

- Three types:
  - **Scaling**: multiple all elements of a row by a nonzero constant.
  - **Replacement**: Replace one row by the sum of itself and a multiple of another row.
  - **Interchange**: Interchange two rows.
- Two matrices are called *row equivalent* if one can be transformed to another through elementary row operations.
- The corresponding systems of linear equations are also *equivalent*, i.e., having the same solutions.

Echelon Form

- A matrix is in *row echelon form* if (REF)
  1. All nonzero rows are above any rows of all zeros, and
  2. Each leading entry of a row (called pivot) is strictly in a column to the right of the leading entry of the row above it.
- These two conditions imply that all entries in a column below a pivot are zeros
- Examples of matrices in row echelon form:

pivot columns

$$\begin{bmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{2} & -8 & 8 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & -3 & 2 & 1 \\ 0 & \boxed{2} & -4 & 8 \\ 0 & 0 & 0 & \boxed{2.5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

non-pivot column

- The marked positions are the *pivot positions*.
- A *pivot column* is a column that contains a pivot.

Reduced Row Echelon Form

RREF

- A matrix is in *reduced row echelon form* if it is in row echelon form, and additionally, it satisfies:
  3. The leading entry in each nonzero row is 1.
  4. Each leading 1 is the only nonzero entry in its column
- Examples of matrices in reduced row echelon form:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 8 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduce a matrix to its echelon form

- Gaussian elimination converts a matrix to an equivalent matrix in echelon form:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boxed{1} & 0 & 0 & 29 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} \boxed{1} & 0 & -5 & 0 \\ 0 & \boxed{1} & -4 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

RREF always exists and is unique

- Any nonzero matrix may be row reduced (i.e., transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations.
- However, each matrix is row equivalent to *one and only one* reduced echelon matrix.

Row reduction algorithm

Reduce a matrix to an echelon form through elementary operations:

1. Begin with the leftmost nonzero column - the first pivot column
2. Select a nonzero entry in the pivot column as a pivot (interchange rows if necessary)
3. Use row replacement to create zeros in positions below the pivot
4. Cover the row containing the pivot position and all rows above it. Repeat steps 1-3 to the remained submatrix.

   $\Longrightarrow$ *row echelon form*

5. Backward phase: Beginning with the rightmost pivot and working upward and to the left,
   - Scale the row containing the pivot to make the leading entry 1
   - Create zeros above the pivot by row replacement

   $\Longrightarrow$ *reduced row echelon form*

Example of row reduction algorithm

- Augmented matrix
  $$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$
- $\xrightarrow{4\times R1+R3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$
- $\xrightarrow{\frac{1}{2}\times R2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$
- $\xrightarrow{3\times R2+R3} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
- This is the row echelon form, now we are going to transform it to reduced row echelon form

Example

- $$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$
- $$\xrightarrow{4\times R3+R2} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$
- $$\xrightarrow{-1\times R3+R1} \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$
- $$\xrightarrow{2\times R2+R1} \begin{bmatrix} \boxed{1} & 0 & 0 & 29 \\ 0 & \boxed{1} & 0 & 16 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$
- This is the reduced row echelon form

Solutions of linear systems

- The row reduction algorithm leads to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.
- Suppose that the augmented matrix of a linear system has been changed into the equivalent reduced echelon form.

$$A \to \begin{array}{cccc} \boxed{x_1 \quad x_2} & & x_3 & \\ \end{array}$$

$$\begin{bmatrix} \boxed{1} & 0 & -5 & 1 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$$

$x_1 = 5x_3 + 1$

$x_2 = -x_3 + 4$

x_1, x_2

- The variables x_1 and x_2, corresponding to pivot columns, are called basic variables. The other variable, x_3, is called a free variable. x_3

Solutions of linear systems

Consider the example: $\begin{bmatrix} \boxed{1} & 0 & -5 & 1 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned}$

- Basic variables: x_1 and x_2, corresponding to pivot columns
- Free variable: x_3
- Key observation: RREF places each basic variable in one and only one equation.
- Solve the reduced system of equations for basic variables in terms of free variables:

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 &\ \textit{is free} \end{aligned}$$

- "x_3 is free" means that it can take any value. For example, $x_1 = 1, x_2 = 4, x_3 = 0$ or $x_1 = -4, x_2 = 5, x_3 = -1$

Parametric descriptions of solution sets

- In the previous example, the solution

$$\begin{aligned} x_1 &= 1 + 5x_3 \\ x_2 &= 4 - x_3 \\ x_3 &\ \textit{is free} \end{aligned}$$

  is a parametric description of the solutions set in which the free variables act as parameters.
- Solving a system amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Existence and uniqueness theorem

- A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column, i.e., if and only if an echelon form of the augmented matrix has no row of the form
$$\begin{bmatrix} 0 & \dots & 0 & b \end{bmatrix}$$
with b nonzero.
- If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Matrix

- A matrix is a rectangular array of numbers, arranged in rows and columns.
- For example:
- $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 4 & -2.5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is called a 2×3 (read two by three) matrix.
- Each entry is referred to by two indexes (i, j), specifying the row and column of the entry in A
- a_{ij} : entry at i-th row and j-th column

Matrix

- In general, an $m \times n$ matrix has m rows and n columns.
- $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$
- a_{ij}: entry at i-th row and j-th column
- In Python numpy, a_{ij} is written $A[i,j]$ with the index starting from 0.

scipy

Matrix equation $Ax = b$

$m \times n$ matrix, or $A \in R^{m \times n}$, $b \in R^{m}$

$x \in R^{n}$

- **Definition**: If A is an $m \times n$ matrix, with columns $a_1, \cdots, a_n$, and if x is in R^n, then the product of A and x, denoted by Ax, is the **linear combination** of the columns of A using the corresponding entries in x as weights; that is

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{= \vec{x}} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_n \vec{a}_n$$

- Ax is defined only if the number of columns of A equals the number of entries in x.

Matrix equation $Ax = b$

- Consider the following system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

- Write it as a **matrix equation**

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Matrix equation $Ax = b$

- If A is an $m \times n$ matrix, with columns $a_1, \cdots, a_n$, and if b is in R^n , then the matrix equation

$$Ax = b$$

has the same solution set as the vector equation

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b,$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & b \end{bmatrix}$$

- The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A.

Computing Ax

- Let A be an $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- Then

$$\vec{a}_1 \qquad \vec{a}_2 \qquad \vec{a}_n$$

$$Ax = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix}$$

-

Dot Product

Let x and y be two vectors in R^n. We define the dot product between two vectors as:

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

Transpose

- Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ Column Vectors

$\boxed{\text{dot product}}$

- Define $y^T = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}$ - turning a column vector into a row vector
- Then

Inner product $\quad x \cdot y = y^T x \quad \rightarrow$ Matrix

$x, y \in \mathbb{R}^n$

↳ matrix multiplication.

y^T: $1 \times n$ matrix

x: $n \times 1$ matrix

Matrix equation: $Ax = 0$

A system of linear equation is said to be **homogeneous** if it can be written in the form $Ax = 0$, where A is an $m \times n$ matrix and 0 is the zero vector in R^m.

- $x = 0$ is always a solution, called the trivial solution.
- $Ax = 0$ has a nontrivial solution (nonzero vector) if and only if the equation has at least one free variable.

- **Definition**: An indexed set of vectors $\{v_1, v_2, \ldots, v_p\}$ in R^n is called **linearly independent** if

$$x_1 v_1 + x_2 v_2 + \ldots + x_p v_p = 0$$

has only the trivial solution. ($x_1 = x_2 = \cdots = x_p = 0$)

- Otherwise, the set is called **linearly dependent**.

$$A = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_p \\ | & | & & | \end{bmatrix}$$

then $v_i = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_p v_p$ for some $\beta_1, \cdots, \beta_p$.

If $Ax = 0$ has only a trivial soln.
then: linearly indep.

Summary statements

Let $\{v_1, v_2, \cdots, v_p\}$ be a set of vectors in R^n, and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_p \end{bmatrix}$, the following statements are equivalent:

a) The set is linearly dependent.

b) $Ax = 0$ has nontrivial solutions.

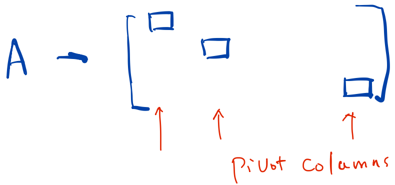
c) A has at least one free variable.

d) The number of pivots in A is less than p.

e) $\text{rank}(A) < p$. Define **rank(A) = number of pivots in A**. = # of pivot columns

= # of basic variables

= $p -$ # of free variables



Summary statements

Let $\{v_1, v_2, \cdots, v_n\}$ be a set of vectors in R^n, and $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$, the following statements are equivalent:

$$x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \vec{0}$$

a) The set is linearly independent.
b) $Ax = 0$ has only trivial solutions.
c) A has no free variables.
d) rank(A) $= n$. Such a matrix is called **non-singular**.
e) $Ax = b$ has exactly one solution.

Matrix

Consider an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathrm{a}_1 & \mathrm{a}_2 & \dots & \mathrm{a}_\mathrm{n} \end{bmatrix}$$

- a_{ij} is the scalar entry in the ith row and jth column, called the (i, j)-entry.
- Each column is a vector in R^m.
- Two matrices are equal if they have the same size and the corresponding entries are equal
- $a_{11}, a_{22}, \dots$ are called the **diagonal entries**
- A is called diagonal if all non-diagonal entries are zero
  - The identity matrix I_n is a square diagonal matrix with diagonal being 1
- The zero matrix is a matrix in which all entries are zero, written as 0.

Matrix operations

Given two $m \times n$ matrices A and B,

- Sum: $A + B$ is an $m \times n$ matrix whose (i, j)-entry is $a_{ij} + b_{ij}$
- Multiplication by a scalar: $rA = Ar$ is an $m \times n$ matrix whose (i, j)-entry is ra_{ij}, where r is a scalar.
- Matrix vector product: $Ax = x_1 a_1 + x_2 a_2 + \ldots + x_n a_n$

Properties of matrix operations

Given A, B, C matrices of the same size, and scalars r and s,

- $(A+B)+C = A+(B+C)$
- $A+B = B+A$
- $A+0 = A$
- $r(A+B) = rA+rB$
- $(r+s)A = rA+sA$
- $r(sA) = (rs)A$

MATRIX MULTIPLICATION

$A: m \times n$

$\underline{AB}$

$B: n \times p$

- **Definition**: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product AB is the $m \times p$ matrix whose columns are $Ab_1, \cdots, Ab_p$.
- That is

$$\boxed{AB} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.

$\underline{m \times p}$

Row—-column rule for computing AB

- Now let's check the (i,j)-entry of AB:

$$\begin{aligned}(AB)_{ij} &= \text{the } i\text{-th entry of the } j\text{-th column}\\ &= \text{the } i\text{-th entry of } Ab_j\\ &= b_j \cdot (\text{the } i\text{-th row of } A)\\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}\end{aligned}$$

- The (i,j)-entry of AB is the sum of the products of corresponding entries from row i of A and column j of B

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$

Special Cases

- An nx1 matrix can be viewed as a vector in R^n (column vector)
- A row vector can be viewed as a 1xn matrix.
- (Dot product) A row vector times a column vector produces a scalar if they are of the same size.

$$\underbrace{\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}}_{1 \times n} \times \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{n \times 1} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

Scalar

$\in R$

row vector × Column Vector → R

- (Out product) A column vector times a row vector produces a matrix.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \times \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_m b_1 & a_m b_2 & \dots & a_m b_n \end{bmatrix} \rightarrow m \times n \text{ matrix}$$

$m \times 1$ $\quad$ $1 \times n$

$\boxed{\text{rank} = 1}$

Column vector × row vector → matrix.

Special cases

- Let A be an mxn matrix,

$$AI_n = A = I_mA$$

-

$$A0 = 0$$

.

I

$0 = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{bmatrix}$

$\vec{0} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

0

Theorems

If the sizes are consistent

- a) $(AB)C = A(BC)$

  associative

- b) $A(B + C) = AB + AC$
- c) $(B + C)A = BA + BC$
- d) $(rA)B = A(rB)$
- e) $I_m A = AI_n$

- $\boxed{AB \neq BA}$ in general. They are not even of the same size!

Not Commutative

Example

Even if they are the same size it is in general not true

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

If $AB = BA$ then A and B are commutable, but in general they are not.

Warnings

- If $AB = AC$ and $A \neq 0$, we cannot conclude $B = C$

Example

Even if they are the same size it is in general not true

$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

But we can clearly see $B \neq C$

Powers of a matrix

Definition: If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A.

$$A^k = A \cdots A \quad (k \text{ times})$$

$A^0 = I$ by convention.

Transpose

- Given an mxn matrix A, the transpose of A is the nxm matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.
- If A $= \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$, then $A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix}$
- $(A^T)_{ij} = a_{ji}$

Properties of matrix transpose

If the sizes are consistent

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(rA)^T = rA^T$
- $(rA)B = A(rB)$
- $(AB)^T = B^T A^T$ (note the reverse order!)

Definition: Let A be an nxn matrix. A is **invertible** if there exists an nxn matrix C such that

$$CA = AC = I_n$$

If A is invertible, we denote C by A^{-1}, and called it the **inverse** of A.

The inverse of 2x2 matrices

A 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if $ad - bc \neq 0$, and in this case its inverse is

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where $det(A) = ad - bc$ is called the determinant of A.

Matrix Determinant

Assign a scalar to each $n \times n$ matrix A, called **det A**. Require it to satisfy three basic properties:

1. $\det(I_n) = 1$
2. The determinant changes sign when two rows are exchanged.
3. The determinant depends linearly on the first row.