Convex set

Definition

A set C is called **convex** if

$$\mathbf{x}, \mathbf{y} \in C \implies \theta \mathbf{x} + (1-\theta)\mathbf{y} \in C \quad \forall \theta \in [0,1]$$

In other words, a set C is convex if the line segment between any two points in C lies in C.

Convex set: examples

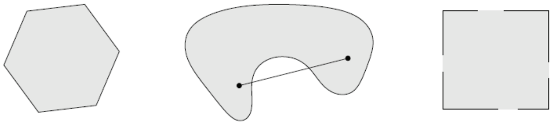

Figure: Examples of convex and nonconvex sets

Convex combination

Definition

A **convex combination** of the points $x_1, \cdots, x_k$ is a point of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k,$$

where $\theta_1 + \cdots + \theta_k = 1$ and $\theta_i \geq 0$ for all $i = 1, \cdots, k$.

A set is convex if and only if it contains every convex combinations of the its points.

Convex hull

Definition

The **convex hull** of a set C, denoted **conv** C, is the set of all convex combinations of points in C:

$$\textbf{conv}\ C = \left\{ \sum_{i=1}^{k} \theta_i x_i \mid x_i \in C, \theta_i \geq 0, i = 1, \cdots, k, \sum_{i=1}^{k} \theta_k = 1 \right\}$$

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Properties:

- A convex hull is always convex
- **conv** C is the smallest convex set that contains C, i.e., $B \supseteq C$ is convex $\implies$ $\textbf{conv}\ C \subseteq B$

Convex hull: examples

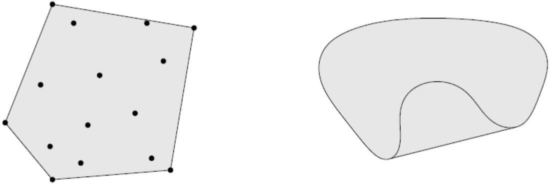

Figure: Examples of convex hulls

Convex cone

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$$x_1, x_2 \in C \implies \theta_1 x_1 + \theta_2 x_2 \in C, \quad \forall \theta_1, \theta_2 \geq 0$$

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The **conic hull** of a set C is the set of all conic combinations of points in C.

Conic hull: examples

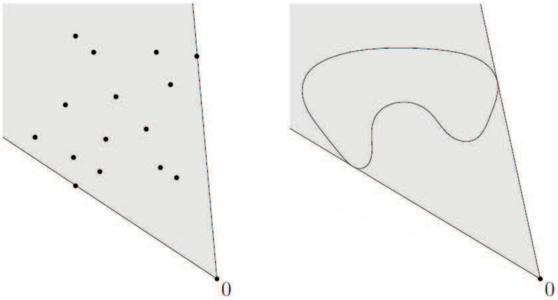

Figure: Examples of conic hull

Hyperplanes and halfspaces

A **hyperplane** is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = b\}$ where $a \neq 0, b \in \mathbb{R}$.

Hyperplanes and halfspaces

A **hyperplane** is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} = b\}$ where $a \neq 0, b \in \mathbb{R}$.

A (closed) **halfspace** is a set of the form $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}^T\mathbf{x} \leq b\}$ where $a \neq 0, b \in \mathbb{R}$.

- **a** is the normal vector
- hyperplanes and halfspaces are convex

Euclidean balls and ellipsoids

Euclidean ball in R^n with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

Euclidean balls and ellipsoids

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ellipsoid in R^n with center x_c:

$$\mathcal{E} = \left\{x \mid (x - x_c)^T P^{-1} (x - x_c) \le 1\right\}$$

where $P \in S^n_{++}$ (i.e., symmetric and positive definite)

- the lengths of the semi-axes of $\mathcal{E}$ are given by $\sqrt{\lambda_i}$, where λ_i are the eigenvalues of P.
- An alternative representation of an ellipsoid: with $A = P^{1/2}$

$$\mathcal{E} = \{x_c + Au \mid \|u\|_2 \le 1\}$$

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Euclidean balls and ellipsoids are convex.

Norms

A function $f : R^n \to R$ is called a **norm**, denoted $\|x\|$, if

- nonegative: $f(x) \geq 0$, for all $x \in R^n$
- definite: $f(x) = 0$ only if $x = 0$
- homogeneous: $f(tx) = |t| f(x)$, for all $x \in R^n$ and $t \in R$
- satisfies the triangle inequality: $f(x + y) \leq f(x) + f(y)$

notation: $\| \cdot \|$ denotes a general norm; $\| \cdot \|_{\mathrm{symb}}$ denotes a specific norm

Distance: $dist(x, y) = \|x - y\|$ between $x, y \in R^n$.

Examples of norms

- ℓ_p-norm on R^n: $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}$
  - ℓ_1-norm: $\|x\|_1 = \sum_i |x_i|$
  - ℓ_∞-norm: $\|x\|_\infty = \max_i |x_i|$
- Quadratic norms: For $P \in S^n_{++}$, define the P-quadratic norm as

$$\|x\|_P = (x^T P x)^{1/2} = \|P^{1/2} x\|_2$$

Equivalence of norms

Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be norms on R^n. Then $\exists \alpha, \beta > 0$ such that $\forall x \in R^n$,

$$\alpha \|x\|_a \leq \|x\|_b \leq \beta \|x\|_a.$$

Norms on any finite-dimensional vector space are equivalent (define the same set of open subsets, the same set of convergent sequences, etc.)

Dual norm

Let $\|\cdot\|$ be a norm on R^n. The associated dual norm, denoted $\|\cdot\|_*$, is defined as

$$\|z\|_* = \sup \ \{z^T x \mid \|x\| \leq 1\}.$$

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- $z^T x \leq \|x\| \; \|z\|_*$ for all $x, z \in R^n$
- $\|x\|_{**} = \|x\|$ for all $x \in R^n$
- The dual of the Euclidean norm is the Euclidean norm (Cauchy-Schwartz inequality)

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- $\|x\|_{**} = \|x\|$ for all $x \in R^n$
- The dual of the Euclidean norm is the Euclidean norm (Cauchy-Schwartz inequality)
- The dual of the ℓ_p-norm is the ℓ_q-norm, where $1/p + 1/q = 1$ (Holder's inequality)
- The dual of the ℓ_∞ norm is the ℓ_1 norm
- The dual of the ℓ_2-norm on $R^{m \times n}$ is the nuclear norm,

$$\begin{aligned}\|Z\|_{2*} &= \sup \ \{tr(Z^T X) \mid \|X\|_2 \leq 1\} \\ &= \sigma_1(Z) + \cdots + \sigma_r(Z) = tr(Z^T Z)^{1/2},\end{aligned}$$

where $r = rank \ Z$.

Norm balls and norm cones

norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $C = \{(x, t) \mid \|x\| \leq t\} \subseteq \mathbb{R}^{n+1}$

- the second-order cone is the norm cone for the Euclidean norm

norm balls and cones are convex

Polyhedra

A **polyhedron** is defined as the solution set of a finite number of linear equalities and inequalities:

$$\mathcal{P} = \{x \mid Ax \preceq b, Cx = d\}$$

where $A \in \mathbb{R}^{m \times n}$, $A \in \mathbb{R}^{p \times n}$, and $\preceq$ denotes *vector inequality* or *componentwise inequality*.

A polyhedron is the intersection of finite number of halfspaces and hyperplanes.

Simplexes

The **simplex** determined by $k+1$ affinely independent points $v_0, \cdots, v_k \in \mathbb{R}^n$ is

$$C = \textbf{conv}\{v_0, \cdots, v_k\} = \left\{ \theta_0 v_0 + \cdots + \theta_k v_k \mid \theta \succeq 0, \mathbf{1}^T\theta = 1 \right\}$$

The affine dimension of this simplex is k, so it is often called k-dimensional simplex in $\mathbb{R}^n$.

Some common simplexes: let $e_1, \cdots, e_n$ be the unit vectors in R^n.

- **unit simplex**: $\textbf{conv}\{0, e_1, \cdots, e_n\} = \{x | x \succeq 0, \mathbf{1}^T\theta \leq 1\}$
- **probability simplex**: $\textbf{conv}\{e_1, \cdots, e_n\} = \{x | x \succeq 0, \mathbf{1}^T\theta = 1\}$

Positive semidefinite cone

notation:

- S^n: the set of symmetric $n \times n$ matrices
- $S_+^n = \{X \in S^n \mid X \succeq 0\}$: symmetric positive semidefinite matrices
- $S_{++}^n = \{X \in S^n \mid X \succ 0\}$ symmetric positive definite matrices

S_+^n is a convex cone, called positive semidefinte cone. S_{++}^n comprise the cone interior; all singular positive semidefinite matrices reside on the cone boundary.

Positive semidefinite cone: example

$$X = \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2 \iff x \geq 0, z \geq 0, xz \geq y^2$$

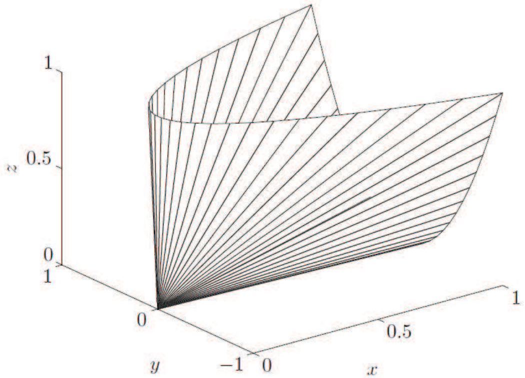

Figure: Positive semidefinite cone: S_+^2

Operations that preserve complexity

- intersection
- affine function
- perspective function
- linear-fractional functions

Intersection

If S_1 and S_2 are convex. then $S_1 \bigcap S_2$ is convex.

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Intersection: example 1

Show that the positive semidefinite cone S_+^n is convex.

Proof.

S_+^n can be expressed as

$$S_+^n = \bigcap_{z \neq 0} \left\{ X \in S^n \mid z^T X z \geq 0 \right\}.$$

Since the set

$$\left\{ X \in S^n \mid z^T X z \geq 0 \right\}$$

is a halfspace in S^n, it is convex. S_+^n is the intersection of an infinite number of halfspaces, so it is convex. □

Intersection: example 2

The set

$$S = \{x \in R^m \mid |\sum_{k=1}^{m} x_k \cos kt| \leq 1, \forall |t| \leq \pi/3\}$$

is convex, since it can be expressed as $S = \bigcap_{|t| \leq \pi/3} S_t$, where $S_t = \{x \in R^m \mid -1 \leq (\cos t, \cdots, \cos mt)^T x \leq 1\}$.

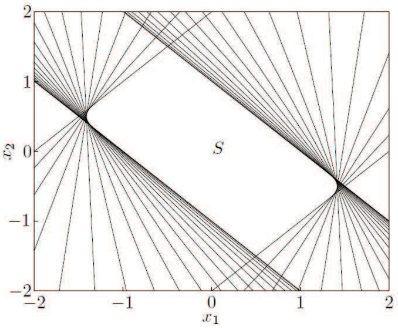

Figure: The set S for $m = 2$.

Affine function

Theorem

Suppose $f : R^n \to R^m$ *is an affine function (i.e.,* $f(x) = Ax + b$*). Then*

- *the image of a convex set under* f *is convex*

$$S \subseteq R^n \text{ is convex} \implies f(S) = \{f(x) \mid x \in S\} \text{ is convex}$$

- *the inverse image of a convex set under* f *is convex*

$$B \subseteq R^m \text{ is convex} \implies f^{-1}(B) = \{x \mid f(x) \in B\} \text{ is convex}$$

Affine function: example 1

Show that the ellipsoid

$$\mathcal{E} = \left\{ x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1 \right\}$$

where $P \in S^n_{++}$ is convex.

Proof.

Let

$$S = \{u \in R^n | \|u\|_2 \leq 1\}$$

denote the unit ball in R^n. Define an affine function

$$f(u) = P^{1/2}u + x_c$$

$\mathcal{E}$ is the image of S under f, so is convex. □

Affine function: example 2

Show that the solution set of linear matrix inequality (LMI)

$$S = \{x \in R^n | x_1 A_1 + \cdots + x_n A_n \succeq B\},$$

where $B, A_i \in S^m$, is convex.

Proof.

Define an affine function $f : R^n \to S^m$ given by

$$f(x) = B - (x_1 A_1 + \cdots + x_n A_n).$$

The solution set S is the inverse image of the positive semidefinite cone S^m_+, so is convex. □

Affine function: example 3

Show that the hyperbolic cone

$$S = \{x \in R^n | x^T P x \leq (c^T x)^2, c^T x \geq 0\},$$

where $P \in S^n_+$, is convex.

Proof.

Define an affine function $f : R^n \to S^{n+1}$ given by

$$f(x) = (P^{1/2}x, c^T x).$$

The S is the inverse image of the second-order cone,

$$\{(z, t) | \|z\|_2 \leq t, t \geq 0\},$$

so is convex. □

Perspective and linear-fractional function

perspective function $P : R^{n+1} \to R^n$:

$$P(x, t) = \frac{x}{t}, \quad \textbf{dom } P = \{(x, t) \mid t > 0\}$$

images and inverse images of convex sets under P are convex.

linear-fractional function $P : R^n \to R^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \quad \textbf{dom } f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under f are convex.

Generalized inequalities: proper cone

Definition

A cone $K \subseteq R^n$ is called a **proper cone** if

- K is convex
- K is closed
- K is solid, which means it has nonempty interior
- K is pointed, which means that it contains no line (i.e., $x \in K, -x \in K \implies x = 0$)

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Examples:

- nonnegative orthant $K = R^n_+ = \{x \in R^n \mid x_i \geq 0, \forall i\}$
- positive semidifinite cone $K = S^n_+$; how about S^n_{++}?
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in R^n \mid x_1 + x_2 t + \cdots + x_n t^{n-1} \geq 0, \forall t \in [0, 1]\}$$

Generalized inequalities: definition

A proper cone K can be used to define a **generalized inequality**, a partial ordering on R^n,

$$x \preceq_K y \Longleftrightarrow y - x \in K \quad x \prec_K y \Longleftrightarrow y - x \in \textbf{int}\, K$$

where the latter is called a strict generalized inequality.

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Examples:

- componentwise inequality ($K = R^n_+$)

$$x \preceq_{R^n_+} y \Longleftrightarrow x_i \leq y_k, \quad \forall i = 1, \cdots, n$$

- matrix inequality ($K = S^n_+$)

$$x \preceq_{S^n_+} y \Longleftrightarrow Y - X \text{ is positive semidefinite}$$

Generalized inequalities: properties

Many properties of $\preceq_K$ are similar to $\leq$ on R:

- transitive: $x \preceq_K y, \; y \preceq_K z \implies x \preceq_K z$
- reflexive: $x \preceq_K x$
- antisymmetric: $x \preceq_K y, \; y \preceq_K x \implies x = y$
- preserved under addition:

$$x \preceq_K y, \; u \preceq_K v \implies x + u \preceq_K y + v$$

- preserved under nonnegative scaling:

$$x \preceq_K y, \; \alpha \geq 0 \implies \alpha x \preceq_K \alpha y$$

- preserved under limits: suppose $\lim x_i = x$, $\lim y_i = y$. Then

$$x_i \preceq_K y_i, \; \forall i \implies x \preceq_K y$$

Minimum and minimal elements

$\preceq_K$ is not in general a linear ordering: we can have $x \not\preceq_K y$ $y \not\preceq_K x$

$x \in S$ is called **the minimum element** of S with respect to $\preceq_K$ if

$$y \in S \implies x \preceq_K y$$

$x \in S$ is called **the minimal element** of S with respect to $\preceq_K$ if

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Example:

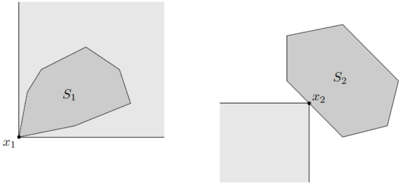

Figure: $K = R^2_+$. x_1 is the minimum element of S_1. x_2 is the minimal element of S_2.

Separating hyperplane theorem

Theorem

Suppose C and D are two convex sets that do not intersect, i.e., $C \cap D = \emptyset$. Then there exist $a \neq 0$ and b such that

$$a^T x \leq b \quad \text{for } x \in C, \qquad \text{and } a^x b \geq b \quad \text{for } x \in D$$

The hyperplane $\{x \mid a^x = b\}$ is called **a separating hyperplane** for C and D.

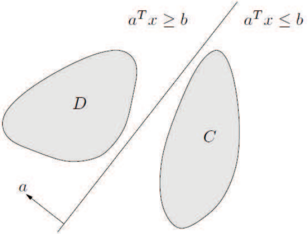

Figure: Examples of convex and nonconvex sets

Supporting hyperplane theorem

supporting hyperplane to set C at boundary point x_0

$$\{x \mid a^x = a^T x_0\}$$

where $a \neq 0$ and satisfies $a^T x \leq a^T x_0$ for all $x \in C$.

Theorem (supporting hyperplane theorem)

If C is convex, then there exists a supporting hyperplane at every boundary point of C.

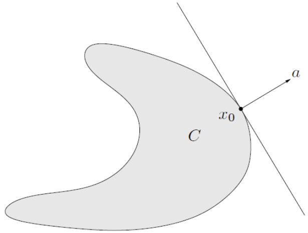

Figure: Examples of convex and nonconvex sets

Dual cones

Definition (dual cones)

Let K be a cone. The set

$$K^* = \{y \mid x^T y \geq 0 \ \forall x \in K\}$$

is called the **dual cone** of K.

Property:

- K^* is always convex, even when the original cone K is not (why? intersection of convex sets)
- $y \in K^*$ if and only if $-y$ is the normal of a hyperplane that supports K at the origin

Dual cones : examples

Examples:

- $K = R_+^n$: $K^* = R_+^n$
- $K = S_+^n$: $K^* = S_+^n$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\| \leq t\}$: $K^* = \{(x,t) \mid \|x\|_* \leq t\}$

the first three examples are self-dual cones

Dual of positive semidefinite cone

Theorem

The positive semidefinite cone S_+^n is self-dual, i.e., given $Y \in S^n$,

$$\mathbf{tr}(XY) \geq 0 \ \ \forall X \in S_+^n \iff Y \in S_+^n$$

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Proof.

To prove $\Longrightarrow$, suppose $Y \notin S_+^n$. Then $\exists q$ with $q^T Y q = \mathbf{tr}(qq^T Y) < 0$, which contradicts the lefthand condition.

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To prove $\Longrightarrow$, suppose $Y \notin S_+^n$. Then $\exists q$ with $q^T Y q = \mathbf{tr}(qq^T Y) < 0$, which contradicts the lefthand condition.

To prove $\Longleftarrow$, since $X \succeq 0$, write $X = \sum_{i=1}^n \lambda_i q_i q_i^T$, where $\lambda_i \geq 0$ for all i. Then

$$\mathbf{tr}(XY) = \mathrm{tr}(Y \sum_{i=1}^n \lambda_i q_i q_i^T) = \sum_{i=1}^n \lambda_i q_i^T Y q_i \geq 0,$$

because $Y \succeq 0$. □

Dual of a norm cone

Theorem

The dual of the cone $K = \{(x, t) \in R^{n+1} \mid \|x\| \leq t\}$ *associated with a norm* $\| \cdot \|$ *in* R^n *is the cone defined by the dual norm,*

$$K^* = \{(u, s) \in R^{n+1} \mid \|u\|_* \leq s\},$$

where the dual norm is given by $\|u\|_* = \sup\{u^T x \mid \|x\| \leq 1\}$.

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Proof.

We need to show

$$x^T u + ts \geq 0 \; \forall \|x\| \leq t \Longleftrightarrow \|u\|_* \leq s$$

The $\Longleftarrow$ direction follows from the definition of the dual norm.

To prove $\Longrightarrow$, suppose $\|u\|_* > s$. Then by the definition of dual norm, $\exists x$ with $\|x\| \leq 1$ and $x^T u \geq s$. Taking $t = 1$, we have $u^T(-x) + v < 0$, which is a contradiction. $\square$

Dual cones and generalized inequalities

Properties of dual cones: let K^* be the dual of a convex cone K.

- K^* is a convex cone (intersection of a set of homogeneous halfspaces)
- $K_1 \subseteq K_2 \implies K_2^* \subseteq K_1^*$
- K^* is closed (intersection of a set of closed sets)
- K^{**} is the closure of K (if K is closed, then $K^{**} = K$)
- dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0$$