Proof for DP argument in Chapter 15

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January 17, 2010

Abstract

KEY WORDS:

We need to know G is almost surely a probability distribution. That amounts to showing that the w_ks are between 0 and 1, which is clear, and that $\sum_{k=1}^{\infty} w_k = 1$ with probability one. Lemma 1 allows us to reformulate the problem.

Lemma 1:

$$\sum_{k=1}^{\infty} w_k \equiv \sum_{k=1}^{\infty} q_k \prod_{j=1}^{k-1}(1-q_j) = 1 - \prod_{k=1}^{\infty}(1-q_k)$$

Proof: Take $r_i \equiv (1-q_k)$,

$$\begin{aligned}
\sum_{k=1}^{\infty} q_k \prod_{j=1}^{k-1}(1-q_j) &= q_1 + q_2(1-q_1) + q_3(1-q_1)(1-q_2) + \cdots \\
&= (1-r_1) + (1-r_2)r_1 + (1-r_3)r_1 r_2 + \cdots \\
&= 1 - r_1 + r_1 - r_2 r_1 + r_1 r_2 - r_3 r_1 r_2 + \cdots \\
&= 1 - \prod_{k=1}^{\infty} r_k \\
&= 1 - \prod_{k=1}^{\infty}(1-q_k)
\end{aligned}$$

By Lemma 1, $\sum_{k=1}^{\infty} w_k = 1$ with probability 1, if an only if $\prod_{k=1}^{\infty}(1-q_k) = 0$ with probability 1. By the symmetry of the problem, it is equivalent to show that $\prod_{k=1}^{\infty} q_k = 0$ with probability 1.

Recall that the q_ks have iid Beta distributions. We proceed under the assumption that $\mathrm{E}[\log(q_1)]$ is finite. It is certainly no greater than 0, so the only other possibility is that $\mathrm{E}[\log(q_1)] = -\infty$. We discuss that possibility at the end.

Clearly the $\log(q_k)$s are iid with finite mean so by the SLLN

$$\frac{\sum_{k=1}^{n} \log(q_k)}{n} \overset{a.s.}{\to} \mathrm{E}[\log(q_1)] < 0.$$

It follows that

$$e^{\sum_{k=1}^{n} \log(q_k)/n} \overset{a.s}{\to} e^{\mathrm{E}[\log(q_1)]} < 1$$

and

$$\left(e^{\sum_{k=1}^{n}\log(q_k)}\right)^{1/n} \stackrel{a.s.}{\longrightarrow} e^{\mathrm{E}[\log(q_1)]} < 1$$

or

$$\left(\prod_{k=1}^{n} q_k\right)^{1/n} \stackrel{a.s.}{\longrightarrow} e^{\mathrm{E}[\log(q_1)]} < 1.$$

Since this occurs with probability one, our result follows from the following lemma about scalar sequences.

Lemma 2: For a scalar sequence x_n defined on $[0,1]$, if $x_n^{1/n} \to p \in [0,1)$ then $x_n \to 0$.

Proof: Since $x_n^{1/n} \to p$, for any δ there exists N_δ s.t. $n \geq N_\delta$ implies $|x_n^{1/n} - p| < \delta$. For any $\epsilon > 0$, pick $\delta > 0$ such that $(p+\delta)^n < \epsilon$ for $n \geq N_\delta$. Since $-\delta < x_n^{1/n} - p < \delta$, $0 \leq x_n^{1/n} < p + \delta$ and $0 \leq x_n < (p+\delta)^n < \epsilon$ for any $n \geq N_\delta$, which completes the proof.

If $\mathrm{E}[\log(q_1)] = -\infty$, we can create truncated versions of the q_ks, say $\tilde{q}_k$s that have finite expectation. Then

$$\frac{\sum_{k=1}^{n}\log(q_k)}{n} \leq \frac{\sum_{k=1}^{n}\log(\tilde{q}_k)}{n} \stackrel{a.s.}{\longrightarrow} \mathrm{E}[\log(\tilde{q}_1)] < 0$$

so

$$0 \leq e^{\sum_{k=1}^{n}\log(q_k)/n} \leq e^{\sum_{k=1}^{n}\log(\tilde{q}_k)/n} \stackrel{a.s}{\longrightarrow} e^{\mathrm{E}[\log(q_1)]} < \epsilon$$

and

$$\left(\prod_{k=1}^{n} q_k\right)^{1/n} \to 0.$$