LAB 8

The Determinant Function

Topics: the determinant function; its properties with respect to row operations; its behavior on nonsingular and singular matrices; a method to compute the value of the determinant function using row operations.

Introduction

We assume that the hand computation of determinants of 2×2 and 3×3 matrices has been discussed using the standard scheme involving products of diagonal arrangements of matrix entries. Figure 1 illustrates the so called 2×2 and 3×3 trick for determinants.

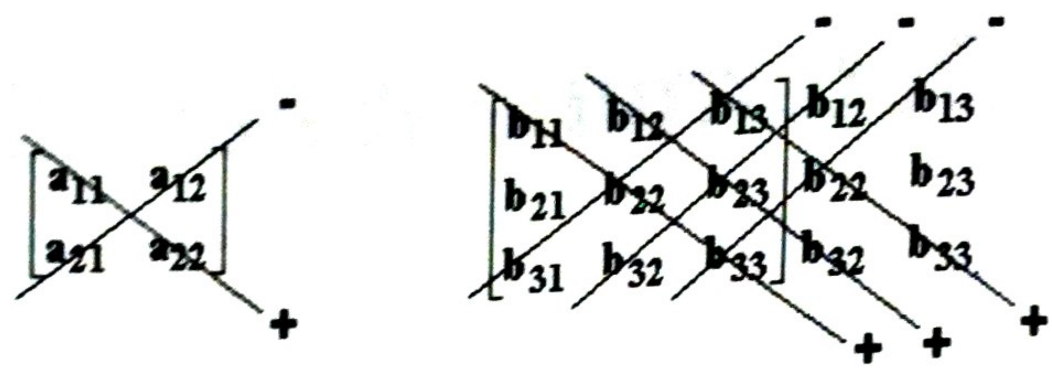

Figure 1.

The products of the entries are computed along the lines on the figure. The products are then added with the sign at the end of the line attached.

Section 8.1

Experiments to Investigate the Determinant Function

The *determinant* is a function from the square matrices of any size to the complex numbers. The function is denoted by **det**. Its value when acting on a square matrix A is denoted by **det** A when we write it and by the command **det**(A) in MATLAB . Some books also use the notation $|\ A\ |$ for the value of the determinant. We will use MATLAB to explore the properties of the determinant function by a series of experiments.

Exercises 8.1

1. Construct a 2×2 matrix with a row of zeros and compute its determinant. Repeat for a 3×3 and a 4×4 matrix with a row of zeros. Construct 2×2, 3×3, and 4×4 matrices each with a column of zeros and compute their determinants.

> Conjecture:
>
> The determinant of a matrix with a row or column of zeros is ______________.

2. Construct a 2×2 matrix with two equal rows and compute its determinant. Repeat the computation for a 3×3 and a 4×4 matrix. Construct 2×2, 3×3, and a 4×4 matrices with two equal columns and compute their determinants.

> Conjecture:
>
> The determinant of a matrix with two equal rows or columns is ______________.

3. The MATLAB command **A = fix(10*rand(3))** generates a 3×3 real matrix $\boldsymbol{A}$. Compute **det(A)** and **det(A')**. Change the 3 in the MATLAB command to several other integer values like 2, 4, 5, 6 and repeat the computations.

> Conjecture:
>
> The determinant of a real matrix and the determinant of its transpose are ______________.

4. Construct a 2×2 diagonal matrix with diagonal entries 5 and 3 and record the value of its determinant. ______________

Construct a 2×2 diagonal matrix with diagonal entries -2 and 9 and record the value of its determinant. ______________

Construct a 3×3 diagonal matrix with diagonal entries 2, 7, and -1 and record the value of its determinant. ______________

Construct a 3×3 diagonal matrix with diagonal entries 4, 0, and 3 and record the value of its determinant. ____________

> **Conjecture:**
>
> The determinant of a diagonal matrix is ______________________.

In MATLAB we can construct a 3×3 upper triangular matrix using the command **A = triu(fix(10*rand(3)))**. Generate several upper triangular matrices of size 3 and compute their determinants. Repeat the procedure for several other sizes.

> **Conjecture:**
>
> The determinant of an upper triangular matrix is ______________________.

5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$. Compute and record **det(A)** = ____________.

We will perform a series of row operations on $\boldsymbol{A}$ and compute the determinant of each new matrix. Always perform the row operation on the original matrix $\boldsymbol{A}$.

> Notation:
>
> - $\boldsymbol{A}_{R_i \leftrightarrow R_j}$ means interchange row i with row j in matrix $\boldsymbol{A}$.
> - $\boldsymbol{A}_{kR_i + R_j}$ means replace row j of $\boldsymbol{A}$ by k times row i plus row j.
> - $\boldsymbol{A}_{kR_i}$ means multiply row i of matrix $\boldsymbol{A}$ by scalar k.

Let $\boldsymbol{B} = \boldsymbol{A}_{R_1 \leftrightarrow R_2}$; **det(B)** = ____________
How is **det(B)** related to **det(A)**? ____________________

Let $\boldsymbol{C} = \boldsymbol{A}_{R_2 \leftrightarrow R_3}$; **det(C)** = ____________
How is **det(C)** related to **det(A)**? ____________________

Let $\boldsymbol{D} = \boldsymbol{A}_{2R_1 + R_2}$; **det(D)** = ____________
How is **det(D)** related to **det(A)**? ____________________

Let $\boldsymbol{E} = \boldsymbol{A}_{-4R_2 + R_3}$; **det(E)** = ____________
How is **det(E)** related to **det(A)**? ____________________

Let $F = A_{3R_1}$; **det(F)** = ____________
How is **det(F)** related to **det(A)**? ________________________

Let $G = A_{-2R_2}$; **det(G)** = ____________
How is **det(G)** related to **det(A)**? ________________________

Let $H = A_{1/2R_3}$; **det(H)** = ____________
How is **det(H)** related to **det(A)**? ________________________

If you have difficulty filling in the following responses repeat the previous experiment with

$$A = \begin{bmatrix} 2 & -5 & 3 \\ 0 & 2 & -1 \\ 3 & 2 & 1 \end{bmatrix}.$$

Conjectures:

If we interchange rows the determinant ________________________.
If we replace one row by a linear combination of itself with another row the determinant ________________________.
If we multiply a row by scalar k the determinant ________________________.

6. Fill in the blanks.

a) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$; **rref(A)** = ____________ **det(A)** = ______

det(rref(A)) = ______

b) Let $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$; **rref(B)** = ____________ **det(B)** = ______

det(rref(B)) = ______

c) Let $C = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 0 \end{bmatrix}$; **rref(C)** = ____________ **det(C)** = ______

det(rref(C)) = ______

d) Let $D = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$; **rref(D)** = ______ **det(D)** = ______

det(rref(D)) = ______

e) <u>True or False:</u> For any square matrix **Q**, **det(Q)** = **det(rref(Q))**. ______

f) Based upon the few experiments in parts a) – d), does there seem to be a connection between the following:

rref is I	det is zero
rref is not I	det is not zero

Draw an arrow between those that appear to be related.

<u>Conjectures: Let **Q** be a square matrix.</u>

If **rref(Q)** = **I**, then **det(Q)** is ______.
If **rref(Q)** ≠ **I**, then **det(Q)** is ______.
The determinant of a nonsingular matrix is ______.
The determinant of a singular matrix is ______.

7. A general way to compute a determinant is to use row operations to reduce it to upper triangular form, keeping track of how the row operations change its value, and then use the fact that the determinant of an upper triangular matrix is the product of the diagonal entries. (See Exercise 4.) To illustrate, let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & 2 \\ 3 & 0 & 4 \end{bmatrix}$. The objective is to compute **det(A)** using properties of the determinant.

Let $B = A_{R_1 \leftrightarrow R_2} = \begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & 1 \\ 3 & 0 & 4 \end{bmatrix}$;

det(B) = ☐ **det(A)** ⟹ **det(A)** = ☐ **det(B)**

Let $C = B_{-2R_1+R_2} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 7 & -3 \\ 3 & 0 & 4 \end{bmatrix}$;

$\mathbf{det(C)} = \square\ \mathbf{det(B)} \Longrightarrow \mathbf{det(A)} = \square\ \mathbf{det(C)}$

Let $D = C_{-3R_1+R_3} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 7 & -3 \\ 0 & 6 & -2 \end{bmatrix}$;

$\mathbf{det(D)} = \square\ \mathbf{det(C)} \Longrightarrow \mathbf{det(A)} = \square\ \mathbf{det(D)}$

Let $E = D_{1/7R_2} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -3/7 \\ 0 & 6 & -2 \end{bmatrix}$;

$\mathbf{det(E)} = \square\ \mathbf{det(D)} \Longrightarrow \mathbf{det(A)} = \square\ \mathbf{det(E)}$

Let $F = E_{-6R_2+R_3} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & -3/7 \\ 0 & 0 & 4/7 \end{bmatrix}$;

$\mathbf{det(F)} = \square\ \mathbf{det(E)} \Longrightarrow \mathbf{det(A)} = \square\ \mathbf{det(F)}$

Now compute **det(A)** from **det(F)**. ____________________
Check your work by computing **det(A)** directly.

8. Follow the procedure in Exercise 7 to compute **det(A)** where $A = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$. Show your work below.

LAB 10

Orthogonal Sets

Topics: orthogonal sets; orthonormal sets; coordinates of a vector relative to an orthonormal basis; projections; construction of an orthonormal basis using the Gram-Schmidt process; command **gschmidt**.

Introduction

Let **V** be an inner product space with the inner product of a pair of vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in **V** denoted by $(\boldsymbol{u}, \boldsymbol{v})$. From Section 9.3 we have that $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal provided $(\boldsymbol{u}, \boldsymbol{v}) = 0$. Here we consider sets of orthogonal vectors and investigate bases which are orthogonal sets.

Let $\mathbf{S} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ be a set of vectors in **V**.

- **S** is called an *orthogonal set* provided $(\boldsymbol{v}_i, \boldsymbol{v}_j) = 0$ for $i \neq j$. (We say that the vectors in **S** are mutually orthogonal.)

- If **S** is an orthogonal set of nonzero vectors, then **S** is linearly independent.

- If **S** is an orthogonal set of nonzero vectors, then

$$\mathbf{T} = \left\{ \frac{\boldsymbol{v}_1}{\| \boldsymbol{v}_1 \|}, \frac{\boldsymbol{v}_2}{\| \boldsymbol{v}_2 \|}, \ldots, \frac{\boldsymbol{v}_k}{\| \boldsymbol{v}_k \|} \right\}$$

  is an orthogonal set in which each vector has length one, where $\| \boldsymbol{v}_j \| = \sqrt{(\boldsymbol{v}_j, \boldsymbol{v}_j)}$. The operation of dividing a nonzero vector by its length is referred to as *normalizing* the vector.

- A set of vectors which is orthogonal and in which each vector has length one is called an *orthonormal set*. (The set **T** above is an orthonormal set.)

Section 10.1 discusses matrices whose columns form an orthonormal basis. Such matrices are called orthogonal and play important roles in a variety of topics.

Section 10.2 develops the projection of one vector onto another and the projection of a vector onto a subspace. We use both a geometric and algebraic approach and show the important role of orthogonal bases. Projections provide an important way to obtain approximations.

Section 10.3 develops an algorithm, the Gram-Schmidt process, for producing an orthonormal basis from an existing basis for a subspace. In effect this shows that the projection techniques from Section 10.2 can always be applied.

Section 10.1

Orthonormal Bases

Let $\mathbf{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ be an orthonormal basis for an inner product space **V**. Then for any vector $\boldsymbol{v}$ in **V** it is easy to compute the coordinates of $\boldsymbol{v}$ relative to **S** using the inner product. Suppose that we want to find $c_i, i = 1, 2, \ldots, n$ such that

$$v = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$$

Then using the property that the vectors in **S** form an orthonormal set of vectors, upon taking the inner product of each side of the previous expression with u_k we have that

$$c_k = (v, u_k), \qquad \text{for } k = 1, 2, ..., n.$$

Now let T be an $n \times n$ matrix whose jth column is denoted by w_j. It is instructive to view T as

$$T = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

The definition of matrix multiplication enables the entries of the matrix $T' * T$ to be written in terms of the inner products (w_i, w_j).

$$T' * T = \begin{bmatrix} w_1' \\ w_2' \\ \vdots \\ w_n' \end{bmatrix} \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix} = \begin{bmatrix} (w_1, w_1) & (w_1, w_2) & \cdots & (w_1, w_n) \\ (w_2, w_1) & (w_2, w_2) & \cdots & (w_2, w_n) \\ \vdots & \vdots & \vdots & \\ (w_n, w_1) & (w_n, w_2) & \cdots & (w_n, w_n) \end{bmatrix}$$

Two results follow immediately from this form.

- The columns of T form an orthogonal set if and only if $T' * T$ is a diagonal matrix.
- The columns of T form an orthonormal set if and only if $T' * T = I_n$.

  The preceding statement is equivalent to the following:

  The columns of T form an orthonormal set if and only if $T^{-1} = T'$.

Square matrices with orthonormal columns are important in a number of areas and arise later in this chapter. We emphasize this with the following terminology.

A square matrix P with real entries is called **orthogonal** provided $P' = P^{-1}$.

Example 1. Let **S** $= \{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Enter these vectors into MATLAB as **v1, v2, v3** respectively. To verify that S is an orthogonal set you can use the definition and check that in MATLAB **dot(v1, v2)**, **dot(v2, v3)**, and **dot(v1, v3)** are all equal to zero, or in MATLAB form the matrix

C = [v1 v2 v3]

and perform the multiplication $C' * C$ to check that the product is equal to the diagonal matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Since $C' * C$ is not equal to the identity matrix, the set **S** is not orthonormal. To normalize set **S**, enter MATLAB commands

t1 = (1/norm(v1))*v1
t2 = (1/norm(v2))*v2
t3 = (1/norm(v3))*v3

The set **T** = **{t1, t2, t3}** is orthonormal. To verify this, construct the matrix A as

A = [t1 t2 t3]

in MATLAB and then compute $A' * A$. You will see the 3×3 identity matrix displayed.

Let $v = [15 \ \ -7 \ \ 7]'$. The coordinates of v relative to basis S are $[v]_S = [4 \ \ 6 \ \ -5]'$, which can be obtained from **rref([C v])** (See Section 7.2 for details on coordinates of a vector relative to a basis.) or from $C\backslash v$. It is possible to obtain $[v]_T$ in a similar manner. However, since T is an orthogonal basis, $[v]_T$ can be computed directly using $[v]_T = [(v, t1) \ \ (v, t2) \ \ (v, t3)]'$.

Exercises 10.1

1. Let $\mathbf{V} = R^3$ with the standard inner product and let

$$\mathbf{S} = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}.$$

In MATLAB u_1 can be entered by typing **t = 1/sqrt(3); u1 = [t; t; t]** and similarly for **u2** and **u3**.

a) Using MATLAB show that **S** is an orthonormal basis for **V**. Write a brief statement indicating your approach.

____

____

b) For $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, find $[v]_S$ ____________________

c) For $v = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$, find $[v]_S$ ____________________

2. Let $\mathbf{V} = R^2$ with the standard inner product and let

$$\mathbf{S} = \{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

be a basis for $\mathbf{V}$.

a) Using MATLAB show that **S** is an orthogonal set. (State how to do this on the line below.)

____

b) Convert the **S**-basis to an orthonormal basis. Call the new basis **T** and display its vectors below.

c) Let $w = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Find the coordinate vector of w relative to the **T**-basis.

$$[w]_T = \rule{5cm}{0.4pt}.$$

d) What is the coordinate vector of w relative to the **S**-basis?

$$[w]_S = \rule{5cm}{0.4pt}.$$

e) How are the coordinate vectors in c) and d) related?

____

3. In MATLAB use command **x = rand** and then form the matrix

$$A = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$$

a) Compute $A' * A$. ______________________

b) The set of columns of A forms an ______________________ set.

c) Matrix A is an ______________________ matrix.

4. Is the set of columns of the matrix generated by the MATLAB command H = hilb(5) an orthogonal set? Explain.

____

5. Let $v_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$. Find a vector v_3 so that the set $S = \{v_1, v_2, v_3\}$ is orthonormal.

6. Let $A = \begin{bmatrix} 1 & -1 & p \\ 1 & 1 & q \\ 0 & 1 & r \end{bmatrix}$.

a) Determine values p, q, and r so that $A'A$ is a diagonal matrix. How many such values are there?

b) If $r = 1$, then find an orthogonal matrix whose columns are scalar multiples of A.

7. Let P be an $n \times n$ orthogonal matrix and x and y be vectors in R^n.

a) Show that $\| Px \| = \| x \|$.

b) Show that the angle between Px and Py is the same as the angle between x and y.

Section 10.2

Projections

Here we investigate the concept of the (orthogonal) projection of one vector onto another from both a geometrical and a computational standpoint. We use an intuitive development based on trigonometric tools in R^2 and extend the procedures to R^3 and beyond. We conclude our discussion with the projection of a vector onto a plane to set the stage for projections onto subspaces in the next section. Both the computational power and the graphics capability of MATLAB will be used to provide a foundation for the important notion of a projection.

In R^2.

A geometric point of view.

The projection of a vector $\boldsymbol{u}$ onto a vector $\boldsymbol{w}$ is obtained by dropping a perpendicular from the tip of $\boldsymbol{u}$ onto $\boldsymbol{w}$. See Figure 1. (If needed we extend $\boldsymbol{w}$. See Figure 2.) Note that the

Figure 1

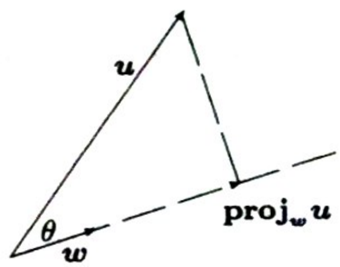

Figure 2

projection in these pictures is in the same direction [1] as $\boldsymbol{w}$ and by using trigonometry the length of the projection is $\cos\theta \parallel \boldsymbol{u} \parallel$. We adopt the following notation. The projection of vector $\boldsymbol{u}$ onto vector $\boldsymbol{w}$ is a vector denoted by

$$\mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u}$$

An algebraic point of view.

We know the length of the projection is $\cos\theta \parallel \boldsymbol{u} \parallel$ and that the projection is a vector in the same direction as $\boldsymbol{w}$. Thus algebraically we can express the projection as its length times a unit vector which has the same direction. Hence we have the expression

$$\mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u} = \cos\theta \parallel \boldsymbol{u} \parallel \frac{\boldsymbol{w}}{\parallel \boldsymbol{w} \parallel}$$

[1] Assume that the angle between $\boldsymbol{u}$ and $\boldsymbol{w}$ is greater than $\pi/2$ radians. Draw the figure and the projection in this case. Explain why in general we say that the projection of vector $\boldsymbol{u}$ onto vector $\boldsymbol{w}$ is a vector parallel to $\boldsymbol{w}$.

Next we use the formula for the cosine of the angle between two vectors (see the Cauchy-Schwarz inequality)

$$\cos\theta = \frac{(\boldsymbol{u}, \boldsymbol{w})}{\| \boldsymbol{u} \| \| \boldsymbol{w} \|}$$

to obtain another formula for the projection. Substituting for $\cos\theta$ we have

$$\textbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u} = \cos\theta \, \| \boldsymbol{u} \| \, \frac{\boldsymbol{w}}{\| \boldsymbol{w} \|} = \frac{(\boldsymbol{u}, \boldsymbol{w})}{\| \boldsymbol{u} \| \| \boldsymbol{w} \|} \, \| \boldsymbol{u} \| \, \frac{\boldsymbol{w}}{\| \boldsymbol{w} \|}$$

Simplifying we have

$$\textbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u} = \frac{(\boldsymbol{u}, \boldsymbol{w})}{\| \boldsymbol{w} \|^2} \boldsymbol{w} \tag{10.1}$$

Example 1. Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$. Find $\textbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u}$.

Using Equation (10.1) for the projection we compute the inner product of vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ and the length of $\boldsymbol{w}$. In MATLAB , command

dot(u,w)

displays

```
ans =

     10
```

and command

norm(w)

displays

```
ans =

    4.1231
```

It follows that $\|\boldsymbol{w}\|^2 = 17$. (In MATLAB use command **ans^2**.) Hence from Equation 10.1,

$$\textbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u} = \frac{10}{17}\boldsymbol{w} = \frac{10}{17}\begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

To see the projection of $\boldsymbol{u}$ onto $\boldsymbol{w}$ use MATLAB command

project(u,w)

The graphics display in two dimensions in routine **project** shows a line drawing similar to Figure 1 with $\mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u}$ labeled by $\boldsymbol{P}$. Note that the line segment perpendicular to $\boldsymbol{w}$ can be written in terms of $\boldsymbol{u}$ and the projection $\boldsymbol{P}$ as $-\boldsymbol{P}+\boldsymbol{u}$. (For more information on routine **project** use **help**.)

A vector space point of view.

In R^2 $\mathbf{span}\{\boldsymbol{w}\}$ is a subspace which geometrically is a line through the origin. The projection of $\boldsymbol{u}$ onto $\boldsymbol{w}$ is a vector in $\mathbf{span}\{\boldsymbol{w}\}$. Since we measure distances as perpendicular distance we see that the vector $\mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u}$ is 'closest' to $\boldsymbol{u}$ since vector $\boldsymbol{s}$ (See Figure 3.) is orthogonal to $\mathbf{span}\{\boldsymbol{w}\}$. Hence the projection of $\boldsymbol{u}$ onto $\boldsymbol{w}$ represents the member of the subspace $\mathbf{span}\{\boldsymbol{w}\}$ that is closest to vector $\boldsymbol{u}$. It also follows from standard vector considerations that we can obtain

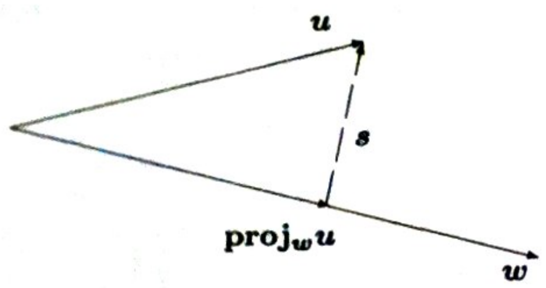

Figure 3

a formula for vector $\boldsymbol{s}$. Vector $\boldsymbol{s}$ is a vector orthogonal to every vector in $\mathbf{span}\{\boldsymbol{w}\}$. We have

$$\boldsymbol{u} \;=\; \mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u} \;+\; \boldsymbol{s}$$

thus

$$\boldsymbol{s} = \boldsymbol{u} - \mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u} \tag{10.2}$$

Example 2. Using the vectors $\boldsymbol{u}$ and $\boldsymbol{w}$ in Example 1, find a vector that is orthogonal to $\mathbf{span}\{\boldsymbol{w}\}$. Using Equation(10.2) we have that a vector $\boldsymbol{s}$ orthogonal to $\mathbf{span}\{\boldsymbol{w}\}$ is

$$\boldsymbol{s} = \boldsymbol{u} - \mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{10}{17}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 11/17 \\ 44/17 \end{bmatrix}$$

To check, compute the inner product of $\boldsymbol{s}$ with any vector in $\mathbf{span}\{\boldsymbol{w}\}$. Here $\mathbf{span}\{\boldsymbol{w}\}$ is any multiple of vector $\boldsymbol{w}$, so let $k\boldsymbol{w}$ represent any vector in $\mathbf{span}\{\boldsymbol{w}\}$.

$$(\boldsymbol{s},k\boldsymbol{w}) = \frac{11}{17}(4k) + \frac{44}{17}(-k) = 0$$

If you used routine **project** in Example 1 (and have not exited MATLAB or done other graphics) type command **figure(gcf)** to redisplay the line drawing of the projection process. Note that $\boldsymbol{s}$, the line segment perpendicular to $\boldsymbol{w}$ in Figure 3, can be used so that $\mathbf{span}\{\boldsymbol{w},\boldsymbol{s}\} = \mathbf{span}\{\boldsymbol{u},\boldsymbol{w}\} = R^2$. Hence set $\{\boldsymbol{w},\boldsymbol{s}\}$ is an orthogonal basis for R^2 while set $\{\boldsymbol{u},\boldsymbol{w}\}$ is a basis, but not an orthogonal set. Press ENTER to return to the command screen, then type the following command.

gtext('s')

Move the cross-hair using the arrow keys to label the vector $\boldsymbol{s}$ which is orthogonal to $\boldsymbol{w}$. Press ENTER to return to the command screen. (Use **help** for more information on **gtext**.)

Summary: Projections give us a way to compute a vector in $\mathbf{span}\{\boldsymbol{w}\}$ that is closest to $\boldsymbol{u}$ and a way to find a vector that is orthogonal to every member of $\mathbf{span}\{\boldsymbol{w}\}$.

These ideas can be extended to R^n and to any real inner product space!

In R^3, the projection of vector $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ onto vector $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ is obtained by employing the identical formula as in Equation (10.1). We use MATLAB and its graphics to illustrate the process.

Example 3. Let $\boldsymbol{u} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 7 \\ 7 \\ -3 \end{bmatrix}$. To determine $\mathbf{proj}_{\boldsymbol{w}}\ \boldsymbol{u}$ use MATLAB command

p = (dot(u,w)/norm(w)^2)*w

To see the components of projection $\boldsymbol{p}$ in rational form type **rational(p)**. To display a line drawing of the process use command

project(u,w)

Use the command **gtext('s')** as described in Example 2 to label vector $\boldsymbol{s}$ on the displayed figure.

To extend the ideas on projections to subspaces of R^3 we proceed as follows. Let $\boldsymbol{u}$ be a vector in R^3 which we want to project onto a plane. A plane is a subspace of dimension 2 in

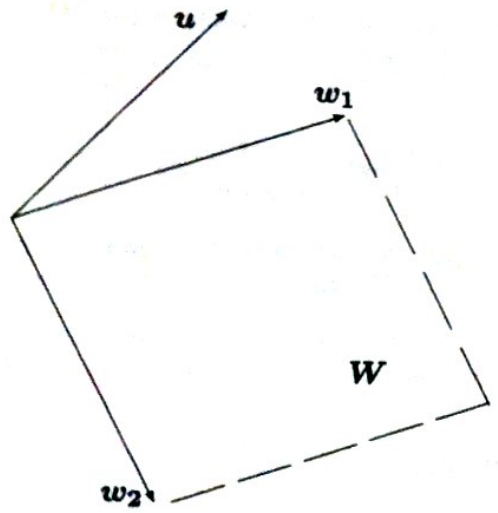

Figure 4

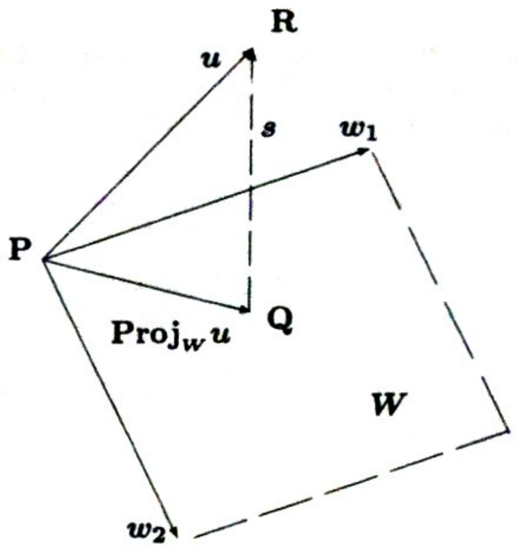

Figure 5

R^3 and is completely defined if we specify a basis. Let $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ be linearly independent **orthogonal vectors** in R^3 so that $\mathbf{W} = \mathbf{span}\{\boldsymbol{w_1}, \boldsymbol{w_2}\}$ and $\boldsymbol{u}$ is not in $\mathbf{W}$. The situation is illustrated in Figure 4. The projection of $\boldsymbol{u}$ onto subspace $\mathbf{W}$ is obtained by dropping a perpendicular from the tip of $\boldsymbol{u}$, denoted by R, until we intersect the plane $\mathbf{W}$ at point Q. In Figure 5 we have indicated the projection by connecting point P with Q. The vector from Q to R is labeled $\boldsymbol{s}$ and is orthogonal to the plane $\mathbf{W}$. Since $\mathbf{W}$ is a subspace, $\boldsymbol{s}$ is orthogonal to $\mathbf{W}$ if and only if it is orthogonal to every vector in the plane $\mathbf{W}$. But $\mathbf{W} = \mathbf{span}\{\boldsymbol{w_1}, \boldsymbol{w_2}\}$, hence $\boldsymbol{s}$ is orthogonal to every linear combination of vectors $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$. Figure 5 provides us

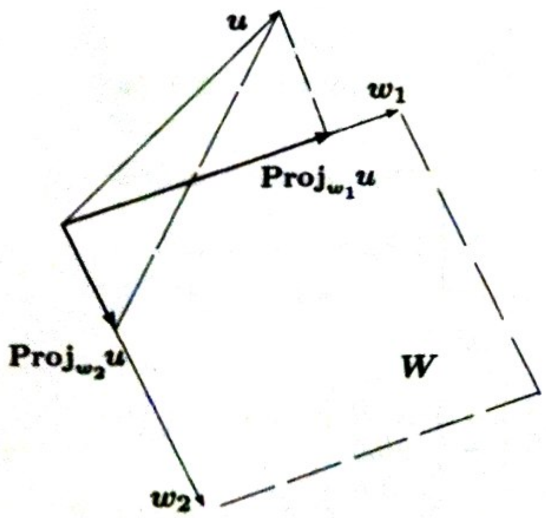

Figure 6

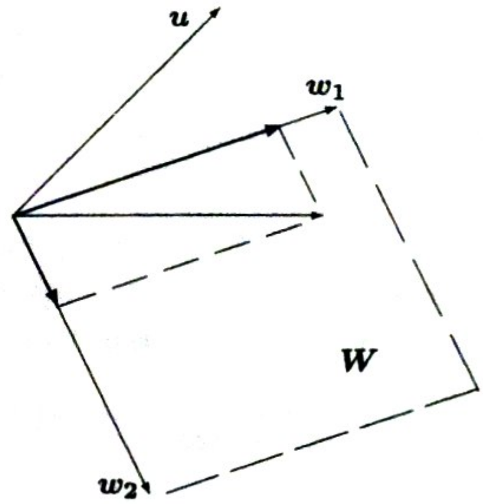

Figure 7

with a picture of the projection $\mathbf{proj}_{\mathbf{W}}\ \boldsymbol{u}$, but to work with the projection we need an algebraic expression for $\mathbf{proj}_{\mathbf{W}}\ \boldsymbol{u}$. Since a basis gives us the minimal amount of information about a subspace it follows that we should consider projecting $\boldsymbol{u}$ onto the basis vectors for $\mathbf{W}$. This is shown in Figure 6. From the previous work in R^2, and the fact that things generalize in a natural fashion, we have

$$\mathbf{proj}_{\boldsymbol{w}_1}\,\boldsymbol{u} = \frac{(\boldsymbol{u},\boldsymbol{w}_1)}{\|\,\boldsymbol{w}_1\,\|^2}\boldsymbol{w}_1 \qquad \text{and} \qquad \mathbf{proj}_{\boldsymbol{w}_2}\,\boldsymbol{u} = \frac{(\boldsymbol{u},\boldsymbol{w}_2)}{\|\,\boldsymbol{w}_2\,\|^2}\boldsymbol{w}_2$$

Since we have an orthogonal basis it can be shown that $\mathbf{proj}_{\mathbf{W}}\,\boldsymbol{u}$ can be written as the sum of the projections of $\boldsymbol{u}$ onto the basis vectors for $\mathbf{W}$. See Figure 7. We have

$$\mathbf{proj}_{\mathbf{W}}\,\boldsymbol{u} = \mathbf{proj}_{\boldsymbol{w}_1}\,\boldsymbol{u} + \mathbf{proj}_{\boldsymbol{w}_2}\,\boldsymbol{u} \tag{10.3}$$

or equivalently

$$\mathbf{proj}_{\mathbf{W}}\,\boldsymbol{u} = \frac{(\boldsymbol{u},\boldsymbol{w}_1)}{\|\,\boldsymbol{w}_1\,\|^2}\boldsymbol{w}_1 + \frac{(\boldsymbol{u},\boldsymbol{w}_2)}{\|\,\boldsymbol{w}_2\,\|^2}\boldsymbol{w}_2$$

Warning: <u>This result is true only when $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are orthogonal.</u>

<u>Example 4.</u> Let $\mathbf{W} = \mathbf{span}\{\boldsymbol{w}_1,\boldsymbol{w}_2\} = \mathbf{span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\begin{bmatrix}-2.5\\1\\.5\end{bmatrix}\right\}$ and $\boldsymbol{u} = \begin{bmatrix}-2\\2\\3\end{bmatrix}$. The set $\{\boldsymbol{w}_1,\boldsymbol{w}_2\}$ is an orthogonal set. Find $\mathbf{proj}_{\mathbf{W}}\,\boldsymbol{u}$ and a vector orthogonal to subspace $\mathbf{W}$. Proceeding as indicated above we have

$$\mathbf{proj}_{\boldsymbol{w}_1}\,\boldsymbol{u} = \frac{(\boldsymbol{u},\boldsymbol{w}_1)}{\|\,\boldsymbol{w}_1\,\|^2}\boldsymbol{w}_1 = \frac{5}{6}\boldsymbol{w}_1$$

$$\mathbf{proj}_{\boldsymbol{w}_2}\,\boldsymbol{u} = \frac{(\boldsymbol{u},\boldsymbol{w}_2)}{\|\,\boldsymbol{w}_2\,\|^2}\boldsymbol{w}_2 = \frac{17}{15}\boldsymbol{w}_2$$

and it follows that from Equation (10.3) that

$$\mathbf{proj}_{\mathbf{W}}\,\boldsymbol{u} = \frac{5}{6}\boldsymbol{w}_1 + \frac{17}{15}\boldsymbol{w}_2 = \begin{bmatrix}-2\\2.8\\1.4\end{bmatrix}$$

From Figure 5 we see that vector $\boldsymbol{s}$ is orthogonal to $\mathbf{W}$ and that

$$\boldsymbol{s} = \boldsymbol{u} - \mathbf{proj}_{\mathbf{W}}\,\boldsymbol{u} = \begin{bmatrix}-2\\2\\3\end{bmatrix} - \begin{bmatrix}-2\\2.8\\1.4\end{bmatrix} = \begin{bmatrix}0\\-0.8\\1.6\end{bmatrix}$$

In Example 4 the projection of $\boldsymbol{u}$ was onto the plane $\mathbf{W} = \mathbf{span}\{\boldsymbol{w}_1,\boldsymbol{w}_2\}$. For illustrative purposes we can think of plane $\mathbf{W}$ as the **xy**-plane. For in fact we can rotate the coordinate system so that $\mathbf{W}$ is rotated into the **xy**-plane. A graphical display of the projection of a vector $\boldsymbol{u}$ onto the **xy**-plane is available in MATLAB . Enter vector $\boldsymbol{u}$ into MATLAB , then type

projxy(u)

Follow the screen directions. Try this with vector $\boldsymbol{u}$ in Example 4.

The key to projections is to use an orthogonal basis.

In fact things are even simpler if we use an orthonormal basis, for then the denominators in Equation (10.3) are 1.

Note: For an orthonormal set of columns $\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_k}$ with $\boldsymbol{W} = \text{span}\{\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_k}\}$, the expression in 10.3 can be computed as follows using matrix $\boldsymbol{M} = [\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_k}]$

$$\textbf{proj}_{\boldsymbol{W}}\boldsymbol{u} = \boldsymbol{M}(\boldsymbol{M}^T\boldsymbol{u}) = \boldsymbol{M}\begin{bmatrix} \boldsymbol{w}_1^T\boldsymbol{u} \\ \boldsymbol{w}_2^T\boldsymbol{u} \\ \vdots \\ \boldsymbol{w}_k^T\boldsymbol{u} \end{bmatrix} = (\boldsymbol{w}_1^T\boldsymbol{u})\boldsymbol{w}_1 + (\boldsymbol{w}_2^T\boldsymbol{u})\boldsymbol{w}_2 + \cdots + (\boldsymbol{w}_k^T\boldsymbol{u})\boldsymbol{w}_k. \quad (10.4)$$

Exercises 10.2

1. Let $\boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a) Find the projection $\boldsymbol{p}$ of $\boldsymbol{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ onto $\boldsymbol{w}$. $\boldsymbol{p} =$ ____________

b) Use command **project(u,w)** to determine the quadrant in which $\boldsymbol{p}$ is located.

Record the quadrant here. ____________

c) Find a scalar k for which the vector $k\boldsymbol{p}$ has a norm that is equal to one.

$k =$ ____________

d) Find a vector $\boldsymbol{s}$ that is orthogonal to $\textbf{span}\{\boldsymbol{w}\}$. $\boldsymbol{s} =$ ____________

2. Let $\boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a) Find the projection $\boldsymbol{p}$ of $\boldsymbol{u} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$ onto $\boldsymbol{w}$. $\boldsymbol{p} =$ ____________

b) Use command **project(u,w)** to determine the quadrant in which $\boldsymbol{p}$ is located.

Record the quadrant here. ______________

c) Find a scalar k for which the vector $k\boldsymbol{p}$ has a norm that is equal to one.

$k =$ ______________

d) Find a vector $\boldsymbol{s}$ that is orthogonal to **span**$\{\boldsymbol{w}\}$. $\boldsymbol{s} =$ ______________

3. Let $\boldsymbol{w} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

a) Find the projection $\boldsymbol{p}$ of $\boldsymbol{u} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ onto $\boldsymbol{w}$. $\boldsymbol{p} =$ ______________

b) Briefly explain your answer to part a). Hint: Use command **project(u,w)**.

____

4. Let $\boldsymbol{w1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\boldsymbol{w2} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$, and $\boldsymbol{u} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$.

a) Find the projection $\boldsymbol{p}$ of $\boldsymbol{u}$ onto **span**$\{\boldsymbol{w1}, \boldsymbol{w2}\}$. $\boldsymbol{p} =$ ______________

b) Find a vector $\boldsymbol{s}$ that is orthogonal to **span**$\{\boldsymbol{w1}, \boldsymbol{w2}\}$. $\boldsymbol{s} =$ ______________

5. Let $\boldsymbol{w1} = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}$, $\boldsymbol{w2} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, and $\boldsymbol{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

a) Find the projection $\boldsymbol{p}$ of $\boldsymbol{u}$ onto $\boldsymbol{W} =$ **span**$\{\boldsymbol{w1}, \boldsymbol{w2}\}$. $\boldsymbol{p} =$ ______________

b) What is the relationship between $\mathbf{proj}_{\boldsymbol{W}}\boldsymbol{u}$ and $\mathbf{proj}_{\boldsymbol{w}_1}\boldsymbol{u}$?
Explain why it occurs.

____

6. Let $\boldsymbol{w1} = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$, $\boldsymbol{w2} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$ and $\boldsymbol{W} =$ **span**$\{\boldsymbol{w1}, \boldsymbol{w2}\}$.

a) Show that $\{\boldsymbol{w1}, \boldsymbol{w2}\}$ is an orthonormal set.

b) Use 10.4 to determine $\mathbf{proj}_{\boldsymbol{W}}\boldsymbol{u}$ where $\boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

7. Let $\boldsymbol{w1} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$, $\boldsymbol{w2} = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ \sqrt{2}/2 \end{bmatrix}$ and $\boldsymbol{W} = \textbf{span}\{\boldsymbol{w1}, \boldsymbol{w2}\}$.

a) Show that $\{\boldsymbol{w1}, \boldsymbol{w2}\}$ is an orthonormal set.

b) Use 10.4 to determine $\textbf{proj}_{\boldsymbol{W}}\boldsymbol{u}$ where $\boldsymbol{u} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$.

Section 10.3

The Gram – Schmidt Process

The ideas about projections in Section 10.2 actually tell us a way to construct an orthonormal basis from an existing basis provided we build the new basis one vector at a time.

The Gram-Schmidt process takes a basis $\mathbf{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n\}$ for a subspace of an inner product space $\mathbf{V}$ and produces a new basis $\mathbf{T} = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ whose vectors form an orthonormal set. The process is often performed in two stages:

- First from the **S**-basis generate a basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ of vectors that are mutually orthogonal. That is, $(\boldsymbol{v}_i, \boldsymbol{v}_j) = 0$, $i \neq j$.

- Second normalize each of the orthogonal basis vectors into a unit vector.

The first stage involves solving a set of equations and the second is easily performed using $\boldsymbol{w}_i = \boldsymbol{v}_i / \| \boldsymbol{v}_i \|$. At each step in the first stage we use projections onto subspaces.

The First Stage

Step 1. Define $\boldsymbol{v}_1 = \boldsymbol{u}_1$.

Step 2. Look for a vector $\boldsymbol{v}_2$ in the $\textbf{span}\{\boldsymbol{v}_1, \boldsymbol{u}_2\}$ that is orthogonal to $\boldsymbol{v}_1$. This will then guarantee that

$$\begin{aligned} \textbf{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2\} &= \textbf{span}\{\boldsymbol{v}_1, \boldsymbol{u}_2\} && \text{since } \boldsymbol{v}_1 = \boldsymbol{u}_1 \\ &= \textbf{span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\} && \text{since } \boldsymbol{v}_2 \text{ is a linear combination of } \boldsymbol{v}_1 \text{ and } \boldsymbol{u}_2 \end{aligned}$$

Let $\boldsymbol{v}_2 = k_1\boldsymbol{v}_1 + k_2\boldsymbol{u}_2$. Find k_1 and k_2 so that $(\boldsymbol{v1}, \boldsymbol{v2}) = \mathbf{0}$.

$$0 = (\boldsymbol{v_1}, \boldsymbol{v_2}) = k_1(\boldsymbol{v_1}, \boldsymbol{v_1}) + k_2(\boldsymbol{v_1}, \boldsymbol{u_2})$$

We have one equation in two unknowns, so let $k_2 = 1$ and solve for k_1. We get

$$k_1 = \frac{-(\boldsymbol{v_1}, \boldsymbol{u_2})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}$$

thus we have

$$\boldsymbol{v_2} = \boldsymbol{u_2} - \frac{(\boldsymbol{v_1}, \boldsymbol{u_2})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}\boldsymbol{v_1} = \boldsymbol{u_2} - \mathbf{proj}_{\boldsymbol{v_1}}\, \boldsymbol{u_2}$$

Step 3. Look for a vector $\boldsymbol{v_3}$ in $\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{u_3}\}$ that is orthogonal to both $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. This will guarantee that $\mathbf{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{u_3}\} = \mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$. Let $\boldsymbol{v_3} = k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + k_3\boldsymbol{u_3}$. Find k_1, k_2, and k_3 so that $(\boldsymbol{v_1}, \boldsymbol{v_3}) = 0$ and $(\boldsymbol{v_2}, \boldsymbol{v_3}) = 0$.

$$\begin{aligned} 0 &= (\boldsymbol{v_1}, \boldsymbol{v_3}) = k_1(\boldsymbol{v_1}, \boldsymbol{v_1}) + k_2(\boldsymbol{v_1}, \boldsymbol{v_2}) + k_3(\boldsymbol{v_1}, \boldsymbol{u_3}) \\ 0 &= (\boldsymbol{v_2}, \boldsymbol{v_3}) = k_1(\boldsymbol{v_2}, \boldsymbol{v_1}) + k_2(\boldsymbol{v_2}, \boldsymbol{v_2}) + k_3(\boldsymbol{v_2}, \boldsymbol{u_3}) \end{aligned}$$

Since by construction $(\boldsymbol{v1}, \boldsymbol{v2}) = 0$ the preceding equations simplify to

$$\begin{aligned} k_1(\boldsymbol{v_1}, \boldsymbol{v_1}) \qquad\qquad &+ k_3(\boldsymbol{v_1}, \boldsymbol{u_3}) = 0 \\ k_2(\boldsymbol{v_2}, \boldsymbol{v_2}) &+ k_3(\boldsymbol{v_2}, \boldsymbol{u_3}) = 0 \end{aligned}$$

Thus we have 2 equations in 3 unknowns. Let $k_3 = 1$, then we find that

$$k_1 = \frac{-(\boldsymbol{v_1}, \boldsymbol{u_3})}{(\boldsymbol{v_1}, \boldsymbol{v_1})} \qquad \text{and} \qquad k_2 = \frac{-(\boldsymbol{v_2}, \boldsymbol{u_3})}{(\boldsymbol{v_2}, \boldsymbol{v_2})}$$

and hence

$$\boldsymbol{v_3} = \boldsymbol{u_3} - \frac{(\boldsymbol{v_1}, \boldsymbol{u_3})}{(\boldsymbol{v_1}, \boldsymbol{v_1})}\boldsymbol{v_1} - \frac{(\boldsymbol{v_2}, \boldsymbol{u_3})}{(\boldsymbol{v_2}, \boldsymbol{v_2})}\boldsymbol{v_2} = \boldsymbol{u_3} - \mathbf{proj}_{\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}\}}\, \boldsymbol{u_3}$$

Other steps: $\boldsymbol{v_k} = \boldsymbol{u_k} - \mathbf{proj}_{\mathbf{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_{k-1}}\}}\, \boldsymbol{u_k}$

The Second Stage

The orthonormal basis for $\mathbf{V}$ is given by

$$\{\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_n}\} = \left\{ \frac{\boldsymbol{v_1}}{\|\boldsymbol{v_1}\|}, \frac{\boldsymbol{v_2}}{\|\boldsymbol{v_2}\|}, \ldots, \frac{\boldsymbol{v_n}}{\|\boldsymbol{v_n}\|} \right\}$$

Example 1. Let $\mathbf{V} = \mathbf{span}\{u_1, u_2, u_3\}$ where

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Use the Gram-Schmidt process to find an orthonormal basis for **V**.

Step 1. Define $v_1 = u_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$.

Step 2. Compute $v_2 = u_2 - \mathbf{proj}_{v_1}\ u_2 = u_2 - \frac{(v_1, u_2)}{(v_1, v_1)} v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix} - \frac{14}{21} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}$.

Step 3. Compute $v_3 = u_3 - \mathbf{proj}_{\mathbf{span}\{v_1, v_2\}}\ u_3 = u_3 - \frac{(v_1, u_3)}{(v_1, v_1)} v_1 - \frac{(v_2, u_3)}{(v_2, v_2)} v_2 =$

$$\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{21} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix} - \frac{(-2/3)}{15/9} \begin{bmatrix} \frac{-1}{3} \\ \frac{-2}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{17}{35} \\ \frac{54}{35} \\ \frac{7}{5} \\ \frac{-22}{35} \end{bmatrix}.$$

The set $\{v_1, v_2, v_3\}$ is an orthogonal basis for **V**. An orthonormal basis is obtained by dividing each vector by it length.

$$w_1 = \frac{v_1}{\sqrt{21}}, \qquad w_2 = \frac{v_2}{\sqrt{17/9}}, \qquad w_3 = \frac{v_3}{\sqrt{6090/1225}}$$

For $\mathbf{V} = R^n$ and the standard inner product both stages of the Gram-Schmidt process are available in MATLAB routine **gschmidt**. Type **help gschmidt** for more details. The following examples illustrate the use of routine **gschmidt**.

Example 2. Let $\mathbf{S} = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ be a basis for R^3. To find an orthonormal basis from **S** using MATLAB enter the vectors u_1, u_2, u_3 as columns of a matrix A and type

B = gschmidt(A)

The display generated is

```
B =

    0.4472    0.7807   -0.4364
         0    0.4880    0.8729
    0.8944   -0.3904    0.2182
```

The columns of B are an orthonormal basis for R^3.

Example 3. We will show how to find an orthonormal basis for R^4 containing scalar multiples of the vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

First enter v_1 and v_2 into MATLAB as vectors **v1** and **v2**, respectively. To find a basis containing scalar multiples of v_1 and v_2, use commands

A = [v1 v2 eye(4)]
rref(A)

The display indicates that the first four columns of A form a basis for R^4. The command **S = A(:,1:4)** produces the matrix with those columns. Type the command

T = gschmidt(S)

The display is

```
T =
    0.5774   -0.3780    0.7237         0
         0    0.3780    0.1974    0.9045
    0.5774    0.7559   -0.0658   -0.3015
   -0.5774    0.3780    0.6580   -0.3015
```

Column 1 of T is $\left(\frac{1}{\|v_1\|}\right) v_1$ and column 2 of T is $\left(\frac{1}{\|v_2\|}\right) v_2$, hence the columns of T form the desired orthonormal basis for R^4.

Explain what to do if **rref(A)** did not indicate that the first four columns of A form a basis for R^4.

Exercises 10.3

1. Let $\mathbf{V} = R^3$ with the standard inner product and let

$$\mathbf{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Use routine **gschmidt** in MATLAB to obtain an orthonormal basis **T** and then find the coordinates of $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ relative to **T**. Record the orthonormal basis and the coordinates of $\boldsymbol{x}$ below.

2. Let $\mathbf{V} = R^4$ with the standard inner product and let

$$\mathbf{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = \left\{ \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Use routine **gschmidt** in MATLAB to obtain an orthonormal basis **T** and then find the coordinates of $\boldsymbol{x} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ relative to **T**. Record the orthonormal basis and the coordinates of $\boldsymbol{x}$ below.

3. Let $\mathbf{V} = R^4$ with the standard inner product and let

$$\mathbf{S} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\} = \left\{ \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix}, \begin{bmatrix} .5 \\ .5 \\ -.5 \\ -.5 \end{bmatrix}, \begin{bmatrix} .5 \\ -.5 \\ -.5 \\ .5 \end{bmatrix}, \begin{bmatrix} .5 \\ -.5 \\ .5 \\ -.5 \end{bmatrix} \right\}.$$

a) Is **S** an orthonormal basis? Circle one: Yes No

Explain your answer.

b) In MATLAB form the matrix $\boldsymbol{T}$ whose columns are the vectors in **S**. Generate a random vector in R^4 using command **x** = **rand(4,1)** and then compute $\| \boldsymbol{x} \|$ and $\| \boldsymbol{T}\boldsymbol{x} \|$. How are the values of the norms related? Repeat the experiment for another arbitrary vector.

____

____

4. Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$. In MATLAB form the matrix **A** = **[v1 v2]** and then use command **gschmidt(A)**. Explain the meaning of the display generated.

____

____

5. Let $\boldsymbol{A} = \begin{bmatrix} 1 & i & 0 \\ i & 0 & 1 \end{bmatrix}$.

a) In MATLAB use command **A′**. Record the result. **A′** = ______________

b) In MATLAB use command **C** = **A′∗A**. Record the result. **C** = ______________

c) What is the relation between $\boldsymbol{C}$ and $\boldsymbol{C}'$?

____

d) Experiment with other complex matrices A to confirm or reject your answer in part c).

Circle one: confirmed not confirmed.

6. A complex matrix A is called Hermitian if it is equal to its conjugate transpose. The command **A′** gives the conjugate transpose in MATLAB .

a) How can you use MATLAB to determine if the matrix A below is Hermitian?

$$A = \begin{bmatrix} 2 & 3-3i \\ 3+3i & 5 \end{bmatrix}$$

____

b) Compute $r = x' * A * x$ for the complex vector below.

$$x = \begin{bmatrix} i \\ 1-i \end{bmatrix} \qquad r = \_\_\_\_\_\_\_\_\_\_\_\_$$

Is r a real number? (Circle one:) YES NO

c) Experiment with other complex vectors x to determine whether $x'Ax$ will always be a real number. (Circle one:)

Always a real number for this matrix A. Not always a real number.

d) Experiment with another Hermitian matrix A and arbitrary vector x to see if $r = x' * A * x$ is always a real number.

(Circle one:) Always a real number. Not always a real number.

7. Let $\mathbf{V} = R^4$ with the standard inner product and let

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \end{bmatrix}.$$

a) Find an orthonormal basis for R^4 containing scalar multiples of the vectors v_1 and v_2. Record your result below.

b) Find an orthonormal basis for R^4 containing scalar multiples of the vectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$, $\boldsymbol{v_3}$. Record your result below.

<< NOTES; COMMENTS; IDEAS >>

LAB 13

The Eigenproblem

Topics: routines **matvec**, **evecsrch** and **mapcirc**; eigenvalues, eigenvectors, characteristic polynomial, roots of the characteristic polynomial; applications.

Introduction

This lab contains both a geometric and an algebraic development of eigen concepts. The geometric development in Section 13.1 uses the function $f(\boldsymbol{x}) = \boldsymbol{Ax}$ where $\boldsymbol{A}$ is a 2×2 matrix and $\boldsymbol{x}$ is a vector in R^2. We compare the input $\boldsymbol{x}$ and output $\boldsymbol{Ax}$ graphically using routine **matvec**. This routine employs MATLAB 's graphical user interface to actively engage the student in experiments which provide a foundation for basic eigen concepts. We follow with routine **evecsrch**, which automates the manual search of **matvec**. Section 13.1 requires only matrix algebra, the notion of length of a vector, and linear independence. It can be used before determinants have been developed. Several exercises explore $f(\boldsymbol{x}) = \boldsymbol{Ax}$ as a linear transformation using the image of the unit circle within another routine **mapcirc**.

Section 13.2 develops the algebraic solution of $f(\boldsymbol{x}) = \boldsymbol{Ax} = \lambda\boldsymbol{x}$ using MATLAB commands for determinants, the characteristic polynomial and its roots, and reduced row echelon form, then concludes with an introduction to MATLAB 's command for eigen computation. This section does not depend on Section 13.1 so it can be used to emphasize the algebraic aspects of the eigen problem. With Section 13.1 this section provides the complement of the geometric exploration.

Section 13.3 provides a set of experiments to explore properties of eigenvalues and eigenvectors, diagonalizable matrices, and applications involving matrix powers, Markov population models, and graph theory applied to a geographical problem involving trade routes.

Section 13.1

Discussion of the General Concept

From a general point of view an eigenvalue is an 'item' in a situation (in mathematics we say problem) that must take on specific values in order for the situation to have a desired outcome. The 'item' (sometimes referred to as a design parameter) is often a constant that can be changed or tuned in an effort to produce a desired outcome (or solution) for the situation. From the description of the situation we must develop an appropriate strategy in order to determine a setting (in mathematics we say value) for the eigenvalue so we can achieve the desired outcome. Hence the determination of an eigenvalue is required as a step of a process to achieve a desired outcome (solution). It follows that we must solve for an eigenvalue to ensure that the desired solution exists. Unfortunately, more than one setting (value) for the eigenvalue could lead to a solution, but not all such settings necessarily produce the outcome we desire.

Consider designing a clock mechanism by using a mass M attached to a spring which is fixed to a support. We desire to have the mass-spring system vibrate so that at 1 second intervals the mass is back at its original position to trigger the movement of a second hand. At time t let the distance of the mass from its original position be denoted by $x(t)$. Then a simple mathematical model for this situation is given by the differential equation

$$\frac{d^2x(t)}{dt^2} + \lambda x(t) = 0$$

where we have constraints $x(0) = 0, x(1) = 0$. (The constraints are called boundary values.) It is known that the eigenvalue λ depends on physical characteristics of the spring and the size of the mass. It happens that there are infinitely many values of λ that will produce a solution for this problem, but not all such values of λ give a solution that we want to use in the design of our clock. For instance, $\lambda = 0$ gives the solution $x(t) = 0$; that is, the mass never moves. Hence the second hand remains fixed. Certainly this choice for the eigenvalue λ gives a valid mathematical solution to the problem, but the corresponding solution is unacceptable if we hope to sell clocks. **An acceptable solution is called an eigenvector (in this case an eigenfunction) associated with the eigenvalue used to obtain that solution.**

Another situation where we vary an 'item' to achieve a result involves tuning in a particular radio station. The tuning knob of a radio varies the capacitance in the tuning circuit. In this way the resonant frequency is changed until it agrees with the frequency of the station we desire. In a broad sense an eigenvalue was selected by the tuning knob to produce a desired solution.

The basic equation that arises to compute eigenvalues λ and corresponding eigenvectors $\boldsymbol{x}$ is

$$f(\boldsymbol{x}) = \lambda \boldsymbol{x}$$

The situation determines the details of the function f. Intuitively this equation says that we seek an *input* $\boldsymbol{x}$ such that *output* $f(\boldsymbol{x})$ is a scalar multiple of $\boldsymbol{x}$. A simple geometric interpretation is that $\boldsymbol{x}$ is an eigenvector of f provided that output $f(\boldsymbol{x})$ is parallel to input $\boldsymbol{x}$. The associated eigenvalue λ is viewed as a magnification factor which affects the direction and length of the output. An easy model of this situation is to consider $f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x}$ in which $\boldsymbol{A}$ is a 2×2 matrix and $\boldsymbol{x}$ is a vector in R^2. In this case the eigen equation is

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

So we seek vectors $\boldsymbol{x}$ so that vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ is parallel to $\boldsymbol{x}$. To illustrate this graphically we use MATLAB routine **matvec**. For a user-chosen 2×2 matrix $\boldsymbol{A}$ use the mouse to choose an input vector $\boldsymbol{x}$ from the unit circle. This input vector is displayed graphically, then the output vector $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ is computed, scaled to have length 1, and displayed graphically. The routine allows multiple selections of inputs $\boldsymbol{x}$ so you can 'home-in' on an eigenvector.

Example 1. In MATLAB type **matvec**. From the menu displayed choose the option for the built-in demo. Your screen should look like Figure 1.

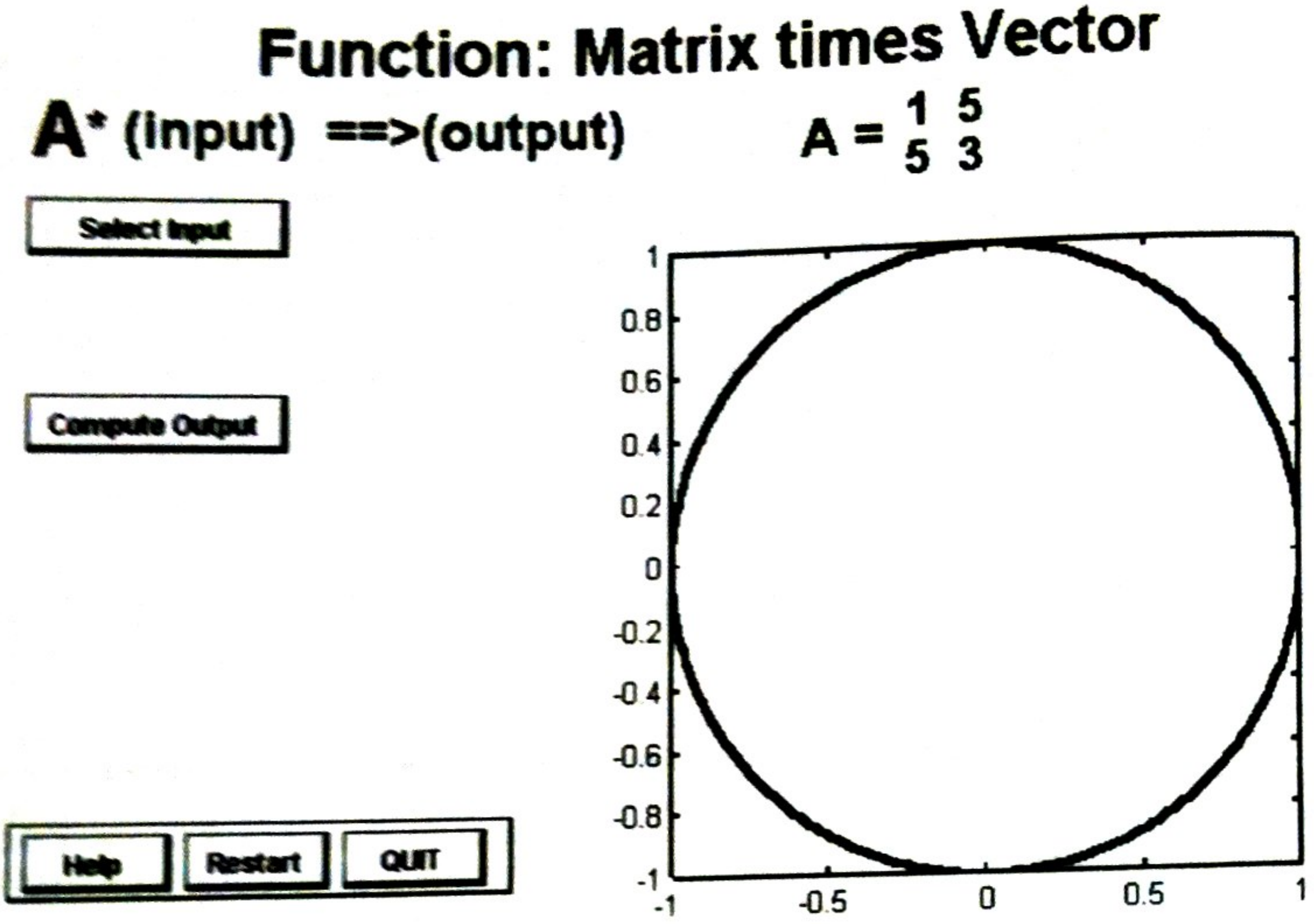

Figure 1.

Click on the button **Select Input**. A message will appear directing you to click on (the circumference of the) circle at the right to select input x. After you make your selection the coordinates of x are displayed and the vector drawn from the center of the unit circle.

Next click on the button **Compute Output**. The coordinates of $y = Ax$ are shown together with the coordinates of the scaled output vector and its graphical representation.

The **More** button, which appears after the execution of the 'Compute Output', encourages further experimentation. Click on this button. Now click once again on 'Select Input'. A small circle remains on the circumference of the unit circle to indicate where previous inputs were chosen.

Use the mouse to choose inputs until you 'home-in' on an eigenvector of matrix A. (Hint: The origin is placed at the center of the circle. Choose inputs in the first quadrant.)

Once you have a close approximation to an eigenvector record the coordinates below.

$$\text{input} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} \quad \text{output} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} \quad \text{scaled output} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

(By close approximation to an eigenvector we mean geometrically that the input and output are nearly parallel and algebraically that the first few decimal places of the coordinates of the input and scaled output are the same.)

Search for a second eigenvector by choosing input vectors in the second quadrant. Once you have a close approximation to an eigenvector record the coordinates below.

$$\text{input} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} \quad \text{output} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} \quad \text{scaled output} = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

To exit **matvec** click on the QUIT button.

In Example 1 matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 5 \\ 5 & 3 \end{bmatrix}$. From the graphical displays you should have observed that for the eigenvector $\boldsymbol{x}_1$ in the first quadrant, the output $\boldsymbol{A}\boldsymbol{x}_1$ was also in the same quadrant. This implies that the corresponding eigenvalue is positive. The numerical value of corresponding eigenvalue λ_1 is obtained from the basic equation $\boldsymbol{A}\boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ by taking the norm of both sides and solving for λ_1;

$$|\lambda_1| = \frac{norm(\boldsymbol{A} * \boldsymbol{x}_1)}{norm(\boldsymbol{x}_1)}$$

But here $\boldsymbol{x}_1$ is a unit vector so the denominator is 1 and we have seen from geometric considerations that this eigenvalue is positive so

$$\lambda_1 = norm(\boldsymbol{A} * \boldsymbol{x}_1)$$

The eigenvector $\boldsymbol{x}_1$ is approximately $\begin{bmatrix} 0.6340 \\ 0.7733 \end{bmatrix}$. Estimate the corresponding eigenvalue λ_1.

Record your work below.

$\lambda_1 =$_______

Let $\boldsymbol{x_2}$ denote the eigenvector in the second quadrant. You should have observed in Example 1 that output $\boldsymbol{Ax_2}$ was in the fourth quadrant. In the space below give an argument to verify that the corresponding eigenvalue λ_2 must be negative.

Eigenvector $\boldsymbol{x_2}$ is approximately $\begin{bmatrix} 0.7733 \\ -0.6340 \end{bmatrix}$. Once again, the command $\lambda_2 = norm(\boldsymbol{A} * \boldsymbol{x_2})$ yields the absolute value of the second eigenvalue. But geometric considerations have shown that this value must be negative. Estimate the corresponding eigenvalue λ_2. Record your work below.

$\lambda_2 =$_______

The routine **matvec** used in Example 1 lets you experiment to determine eigenvectors of a 2×2 matrix. By clicking on the unit circle you really chose a direction for the input vector $\boldsymbol{x}$. The calculation of the output vector $\boldsymbol{y} = \boldsymbol{Ax}$ determines another direction. We say we have an *eigen direction* of $\boldsymbol{A}$ provided the input and output directions are parallel. That is, input and output are in exactly the same direction or in exactly opposite directions. The use of the unit circle is a convenience, really any circle centered at the origin could be used. To emphasize this, in the space below give an argument that verifies that if $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$, that is $\boldsymbol{Ax} = \lambda\boldsymbol{x}$, then $k*\boldsymbol{x}$ is an eigenvector for any scalar $k \neq 0$.

(The preceding shows that the set of eigenvectors of $\boldsymbol{A}$ corresponding to an eigenvalue λ is closed under scalar multiplication.) Thus it is the *'direction'* that is important for an eigenvector, not its length. That is why we could scale the output in **matvec** and display it on the unit circle.

For a 2×2 matrix $\boldsymbol{A}$ the search for eigenvectors can be illustrated by selecting vectors that encompass directions around the unit circle and checking to see if the *input* and *output* = $\boldsymbol{A}$***input* are parallel. Routine **evecsrch** automates this process. The search starts by selecting an input at a randomly chosen point on the unit circle and graphing the corresponding radius.

Next the output is computed, scaled to the unit circle, and graphed. If the input and output are parallel the images are retained, otherwise both are erased. When an eigenvector is detected its components are displayed. The routine stops when it completes the search of the 'entire' unit circle.

In MATLAB type **evecsrch** and follow the directions. Use **evecsrch** to determine the eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 0 & 6 \\ 1 & -1 \end{bmatrix}$. Record the eigenvectors below.

By observation, explain why these eigenvectors are linearly independent.

Determine the corresponding eigenvalues. Show your work below.

In the space below explain why the radii drawn form an **'X'** on the unit circle in the output from **evecsrch**.

Use **evecsrch** to investigate the eigenvectors of $\boldsymbol{B} = \begin{bmatrix} 6 & 7 \\ 0 & 6 \end{bmatrix}$ and determine the corresponding eigenvalues. In the space below summarize your observations and record your calculations.

Next compare the results of **evecsrch** for the preceding matrices A and B in terms of eigenvector information. What is different? Put your discussion below.

Finally, use **evecsrch** to search for the eigenvectors of $A = \begin{bmatrix} 1 & 5 \\ 5 & 3 \end{bmatrix}$, which is the matrix from Example 1. Verify that the eigenvectors are the ones that were computed there.

Exercises 13.1

1. Let $A = \begin{bmatrix} 6 & 1 \\ -2 & 0 \end{bmatrix}$.

   a) Use routine **evecsrch** to approximate eigenvectors x_1 and x_2 of A.

   $$x_1 = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} \qquad x_2 = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

   b) Use routine **matvec** to determine the output when the input is selected close to x_1.

   Is the output in the same direction as x_1, or in the opposite direction? ________

   c) From part b it follows that the eigenvalue λ_1 corresponding to x_1 is (circle one):

   Positive   Negative

   d) Combine parts a and c with command **norm(A*x₁)** to approximate λ_1.

   $\lambda_1 \approx$ ________

e) Use routine **matvec** to determine the output when the input is selected close to $\boldsymbol{x}_2$.

Is the output in the same direction as $\boldsymbol{x}_2$, or in the opposite direction? ____________

f) Combine parts a and e with command **norm(A*x$_2$)** to approximate λ_2.

$\lambda_2 \approx$ ____________

g) Use MATLAB to determine whether $\boldsymbol{A} * \boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ and $\boldsymbol{A} * \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$. Explain any discrepancies in the space below.

2. Let $\boldsymbol{A} = \begin{bmatrix} 1 & 9 \\ 1 & 1 \end{bmatrix}$.

a) Use routine **evecsrch** to approximate eigenvectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ of $\boldsymbol{A}$.

$$\boldsymbol{x}_1 = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix} \qquad \boldsymbol{x}_2 = \begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

b) Use routine **matvec** to determine the output when the input is selected close to $\boldsymbol{x}_1$ that lies in the third quadrant.

Is the output in the same direction as $\boldsymbol{x}_1$ or in the opposite direction? ____________

c) Combine parts a and b with command **norm(A*x$_1$)** to approximate λ_1.

$\lambda_1 \approx$ ____________

d) Use routine **matvec** to determine the output when the input is selected close to $\boldsymbol{x}_2$.

Is the output in the same direction as $\boldsymbol{x}_2$, or in the opposite direction? ____________

e) Combine parts a and d with command **norm(A*x$_2$)** to approximate λ_2.

$\lambda_2 \approx$ ____________

f) Use MATLAB to determine whether $\boldsymbol{A} * \boldsymbol{x}_1 = \lambda_1 \boldsymbol{x}_1$ and $\boldsymbol{A} * \boldsymbol{x}_2 = \lambda_2 \boldsymbol{x}_2$. Explain any discrepancies in the space below.

3. Use routine **matvec** to approximate an eigenvector of $A = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$ that is in the first quadrant. Record your approximate eigenvector $\begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$.

4. Use the matrix A from Exercise 3.

a) Algebraically find an eigenvector $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ by solving the matrix equation $A\boldsymbol{x} = 2\boldsymbol{x}$ for x_1 and x_2. Show your work in the space below.

b) How many solutions were there to the matrix equation $A\boldsymbol{x} = 2\boldsymbol{x}$ in part a?

____

c) Take your solution from part a and scale it to have length 1. Record that vector below.

$$\begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

d) Find another eigenvector of length 1 which is parallel to your solution in part c. Record that vector below.

$$\begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

e) What solution from part a cannot be scaled to have length 1? Explain.

5. Let $A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}$.

a) Use **matvec** to approximate an eigenvector of A that is in the first quadrant. Record that vector below. (Make selections until at least the first two decimal places in the input and scaled output vectors agree.)

$$\begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

b) For this matrix there is a second eigenvector that is orthogonal to the one you found in part a. Determine this vector using **matvec**. Record that vector below. (Make selections until at least the first two decimal places in the input and scaled output vectors agree.)

$$\begin{bmatrix} \_\_\_\_ \\ \_\_\_\_ \end{bmatrix}$$

Verify that the vector in part a is orthogonal to the vector you found here. (Actually it may only be close to orthogonal with the vector in part a since we are using agreement in only the first two decimal places to indicate the input and scaled output vectors are parallel.)

6. Let $\boldsymbol{A}$ be a 2×2 diagonal matrix. Perform a set of experiments using routine **evecsrch** to determine the eigenvectors and corresponding eigenvalues of $\boldsymbol{A}$. Below each of the following matrices record your findings.

$$\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} -8 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Perform additional experiments to formulate a conjecture that describes the eigenvalues and corresponding eigenvectors of a 2×2 diagonal matrix.

Conjecture: ____

____

7. Let $\boldsymbol{A}$ be a 2×2 upper triangular matrix . Perform a set of experiments using routine **evecsrch** to determine the eigenvectors and corresponding eigenvalues of $\boldsymbol{A}$. Below each of the following matrices record your findings.

$$\begin{bmatrix} -5 & 4 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 8 & 1 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 3 \\ 0 & 4 \end{bmatrix}$$

Perform additional experiments to formulate a conjecture that describes the eigenvalues and corresponding eigenvectors of a 2 × 2 upper triangular matrix.

Conjecture: ______________________________

____

8. Let A be a 2 × 2 lower triangular matrix . Perform a set of experiments using routine **evecsrch** to determine the eigenvectors and corresponding eigenvalues of A. Below each of the following matrices record your findings.

$$\begin{bmatrix} -7 & 0 \\ 6 & 2 \end{bmatrix} \qquad \begin{bmatrix} -9 & 0 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$$

Perform additional experiments to formulate a conjecture that describes the eigenvalues and corresponding eigenvectors of a 2 × 2 lower triangular matrix.

Conjecture: ______________________________

____

9. Let A be a 2 × 2 matrix with a zero row. Perform a set of experiments using routine **evecsrch** to determine the eigenvectors and corresponding eigenvalues of A. Below each of the following matrices record your findings.

$$\begin{bmatrix} -5 & 4 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 6 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 3 \\ 0 & 0 \end{bmatrix}$$

Perform additional experiments to formulate a conjecture that describes the eigenvalues and corresponding eigenvectors of a 2×2 matrix with a zero row.

Conjecture: ______________________________

____

10. Let A be a 2×2 matrix with a zero column. Perform a set of experiments using routine **evecsrch** to determine the eigenvectors and corresponding eigenvalues of A. Below each of the following matrices record your findings.

$$\begin{bmatrix} -5 & 0 \\ 6 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 9 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix}$$

Perform additional experiments to formulate a conjecture that describes the eigenvalues and corresponding eigenvectors of a 2×2 matrix with a zero column.

Conjecture: ______________________________

____

11. Let A be a 2×2 symmetric matrix. Perform a set of experiments using routine **evecsrch** to determine an algebraic relationship between the eigenvectors of A. Record your findings below each of the following matrices.

$$\begin{bmatrix} -5 & 9 \\ 9 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -2 \\ -2 & 1 \end{bmatrix} \qquad \begin{bmatrix} -4 & 6 \\ 6 & 0 \end{bmatrix}$$

Perform additional experiments to formulate a conjecture that describes this algebraic relationship between the eigenvectors of a 2×2 symmetric matrix.

Conjecture: ______________________________

____

Formulate a geometric analog of the algebraic relationship.

Geometric formulation: ______________________________

____

In **matvec** we selected input vectors $\boldsymbol{x}$ from the unit circle and displayed their image $\boldsymbol{y} = \boldsymbol{A} * \boldsymbol{x}$ scaled to the unit circle. In **evecsrch** we searched for inputs around the unit circle so that the images would be parallel to the input vector. In Exercises 12 - 19 we look at the entire set of images of the unit circle. We investigate the image of the unit circle geometrically and provide experiments to investigate properties of this image for certain types of 2×2 matrices.

Before performing the investigations below execute routine **mapcirc** which is our primary investigation tool. In MATLAB type **mapcirc**. From the first menu select the built-in demonstration. From the second menu select *not* to see eigenvector information. In the display generated by **mapcirc** the left graph shows the unit circle and the right graph its image under the mapping (or transformation) by matrix $\boldsymbol{A}$. As vectors $\boldsymbol{x}$ are selected from the unit circle on the left their image $\boldsymbol{A} * \boldsymbol{x}$ is computed and displayed on the right graph. For more information on **mapcirc** use **help**.

12. Use **mapcirc** to obtain the image of the unit circle for each of the following matrices. Below each matrix record a sketch of the image and give a geometric description on the line provided.

$$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

______________ ______________ ______________

$$\begin{bmatrix} -1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \qquad \begin{bmatrix} 4.5 & 0 \\ 0 & 1 \end{bmatrix}$$

______________ ______________ ______________

Based on the previous experiments form a conjecture about the shape of the image of the unit circle when $\boldsymbol{A}$ has the following forms.

For $\boldsymbol{A} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ Conjecture: The image of the unit circle is

____

For $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}$ Conjecture: The image of the unit circle is

____

For $\boldsymbol{A} = \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix}$ Conjecture: The image of the unit circle is

____

13. Use **mapcirc** to obtain the image of the unit circle for each of the following matrices. Below each matrix record a sketch of the image and give a geometric description on the line provided.

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

__________ __________ __________

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$

__________ __________ __________

Based on the previous experiments form a conjecture about the shape of the image of the unit circle when $\boldsymbol{A}$ has the following forms.

For $\boldsymbol{A} = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ with $k > 0$ and large Conjecture: The image of the unit circle is

____

For $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ with $k > 0$ and large Conjecture: The image of the unit circle is

____

14. This exercise continues the type of investigation begun in the preceding exercise.

a) Design a set of experiments to investigate the behavior of the images of the unit circle when $\boldsymbol{A}$ has the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ with $k < 0$ and $|\,k\,|$ large. Summarize the behavior in a short paragraph below.

How does the behavior here differ from the case in the preceding exercise where $k > 0$? (Be specific.)

____

b) Design a set of experiments to investigate the behavior of the images of the unit circle when $\boldsymbol{A}$ has the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$ with $k < 0$ and $|\,k\,|$ large. Summarize the behavior in a short paragraph below.

How does the behavior here differ from the case in the preceding exercise where $k > 0$? (Be specific.)

____

15. Use **mapcirc** to obtain the image of the unit circle for each of the following matrices. Below each matrix record a sketch of the image and give a geometric description on the line provided.

$$\begin{bmatrix} 5 & 1 \\ 5 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}$$

__________ __________ __________ __________

From the geometric description above, what is the dimension of the null space of each of these matrices?

____

Each of these matrices is (circle one) singular nonsingular

Conjecture: The image of the unit circle by a _______________ matrix is a

_______________.

If $\boldsymbol{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ what is the image of the unit circle?

____

Is your conjecture immediately above still true? Revise it if necessary.

16. Design a set of experiments to develop a conjecture concerning the image of the unit circle if $\boldsymbol{A}$ is a 2×2 nonsingular matrix.

Conjecture:____

17. Based on the previous exercises, complete the following sentence:
The image of a unit circle under a 2×2 matrix is a __________, a __________, or a __________.

18. For each of the following 2×2 symmetric matrices use **mapcirc** to graphically determine the eigenvectors from their images. On the lines below each matrix first describe the angle between the eigenvectors and on the second line the angle between their images.

$$\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 5 \\ 5 & 0 \end{bmatrix} \qquad \begin{bmatrix} -2 & -1 \\ -1 & 4 \end{bmatrix}$$

__________ __________ __________

__________ __________ __________

Complete the following conjectures:

a) The eigenvectors of a symmetric matrix are ____________________.

b) The images of the eigenvectors of a symmetric matrix are ____________________.

c) The images of the eigenvectors of a symmetric matrix form the ____________________

and ____________________ of the elliptical image.

19. For each of the following symmetric matrices use **mapcirc** to determine the images of the eigenvectors. When **mapcirc** is over return to the command screen and type the following commands. (Choose the option to display the eigenvectors.)

pts = ginput(2); image=pts'

You will be returned to the graphics screen generated by **mapcirc**. The mouse pointer symbol will be a plus sign indicating that you can collect information about points. *Carefully* position the plus sign at the end points of the images of the eigenvectors of ***A*** which appear in a contrasting color. Click the mouse to record the coordinates of those points. After the second click press ENTER to return to the command screen. The coordinates of the end points of the vectors you clicked on will be the columns displayed in matrix **image**. (If you feel you did not position the mouse correctly for the preceding measurements, just repeat the commands above.)

Compute the length of each column of **image** using the following commands.

L1 = norm(image(:,1)),L2=norm(image(:,2))

Record these values on the lines provided and then record the closest integer to the value.

a) $A = \begin{bmatrix} 2.6 & -0.8 \\ -0.8 & 1.4 \end{bmatrix}$ L1 = ______ L2 = ______

b) $A = \begin{bmatrix} -3.6 & 2.8 \\ 2.8 & -0.6 \end{bmatrix}$ L1 = ______ L2 = ______

c) $A = \begin{bmatrix} 3.12 & 3.84 \\ -3.84 & 0.88 \end{bmatrix}$ L1 = ______ L2 = ______

Form a conjecture that describes the values of L1 and L2. (Hint: review Example 1 and the discussion following it.)

Conjecture: ______

What do the lengths L1 and L2 represent geometrically in terms of the image of the unit circle which is an ellipse?

Conjecture: ______

Section 13.2

The Matrix Eigenproblem

We study a surprisingly simple problem involving eigenvalues and eigenvectors that extends the 2×2 matrices considered in Section 1. Given an $n \times n$ matrix $\boldsymbol{A}$, determine how to select vectors $\boldsymbol{x}$ in R^n so that $\boldsymbol{Ax}$ is parallel to $\boldsymbol{x}$. In the context of linear transformations, let $\mathrm{L}: \mathrm{R}^n \to \mathrm{R}^n$ be defined by

$$\mathrm{L}(\boldsymbol{x}) = \boldsymbol{Ax}.$$

We seek input vectors $\boldsymbol{x}$ that are parallel to the output vector $\boldsymbol{Ax}$. To determine a mathematical model for this problem, recall that two vectors are parallel provided they are scalar multiples of one another. Thus our objective is to find $\boldsymbol{x}$ in R^n so that

$$\mathrm{L}(\boldsymbol{x}) = \boldsymbol{Ax} = \lambda \boldsymbol{x}$$

where λ is some scalar. In the matrix equation $\boldsymbol{Ax} = \lambda\boldsymbol{x}$, both the vector $\boldsymbol{x}$ and the scalar λ are unknown. By observation we see that one solution is $\boldsymbol{x} = \boldsymbol{0}$ and then λ could be any value. This solution is uninteresting since $\boldsymbol{A0} = \boldsymbol{0}$ implies only that the zero vector is parallel to itself. Thus we exclude $\boldsymbol{x} = \boldsymbol{0}$ from acceptable solutions. We state our problem as follows.

The Eigenproblem for an $n \times n$ matrix $\boldsymbol{A}$

Determine a nonzero vector $\boldsymbol{x}$ in R^n and scalar λ so that $\boldsymbol{Ax} = \lambda\boldsymbol{x}$. We say λ is an **eigenvalue** of matrix $\boldsymbol{A}$ and $\boldsymbol{x}$ is an associated **eigenvector**.

A basic strategy to compute eigenvalues and eigenvectors of a matrix $\boldsymbol{A}$ begins with the matrix equation $\boldsymbol{Ax} = \lambda\boldsymbol{x}$ and uses concepts we studied previously. We have the following set of equivalent expressions:

$$\boldsymbol{Ax} = \lambda\boldsymbol{x} \Longleftrightarrow \boldsymbol{Ax} = \lambda\boldsymbol{I}_n\boldsymbol{x} \Longleftrightarrow \boldsymbol{Ax}-\lambda\boldsymbol{I}_n\boldsymbol{x} = \boldsymbol{0} \Longleftrightarrow (\boldsymbol{A}-\lambda\boldsymbol{I}_n)\boldsymbol{x} = \boldsymbol{0}$$

Thus our eigenproblem has been recast as a homogeneous system of equations $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$. We seek $\boldsymbol{x} \neq \boldsymbol{0}$ that solves this homogeneous system. However, a square homogeneous system has a nontrivial solution if and only if its coefficient matrix is singular. Matrix $\boldsymbol{A}-\lambda\boldsymbol{I}$ is singular if and only if $\boldsymbol{det}(\boldsymbol{A}-\lambda\boldsymbol{I}) = 0$. Thus eigenvalue λ is viewed as a tuning parameter to force matrix $\boldsymbol{A}-\lambda\boldsymbol{I}$ to be singular. It follows that with such values of λ we are able to determine nonzero vectors $\boldsymbol{x}$ so that $\boldsymbol{Ax} = \lambda\boldsymbol{x}$.

The expression $\boldsymbol{det}(\boldsymbol{A}-\lambda\boldsymbol{I})$ gives a polynomial of degree n in parameter λ which we call the characteristic polynomial of matrix $\boldsymbol{A}$. The expression $\boldsymbol{det}(\boldsymbol{A}-\lambda\boldsymbol{I}) = 0$ is called the characteristic equation of matrix $\boldsymbol{A}$.

The eigenvalues of $\boldsymbol{A}$ are solutions (roots) of the characteristic equation. The corresponding eigenvectors are the solutions of the homogeneous system $(\boldsymbol{A}-\lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$.

Computationally we find the roots of the characteristic polynomial to determine the eigenvalues and then find the general solution of the corresponding homogeneous systems to find the eigenvectors. Hence the solution of the eigenproblem for matrix $\boldsymbol{A}$ is done in two steps.

In MATLAB , once matrix $\boldsymbol{A}$ is entered, we first find the characteristic polynomial of $\boldsymbol{A}$. Command

poly(A)

gives a vector containing the coefficients of the characteristic polynomial with the coefficient of the highest power term displayed first and the constant term last. (Zeros are used for the coefficient of any power of λ that is explicitly missing.) Command

roots(poly(A))

gives a vector containing the roots of the characteristic polynomial, that is the eigenvalues of $\boldsymbol{A}$. We illustrate these commands in the following example.

Example 1. Let $A = \begin{bmatrix} 5 & -8 & -1 \\ 4 & -7 & -4 \\ 0 & 0 & 4 \end{bmatrix}$. Enter A into MATLAB . Then command

c = poly(A)

displays

```
c =

     1    -2   -11    12
```

which implies that the characteristic polynomial of A is

$$1\lambda^3 - 2\lambda^2 - 11\lambda + 12$$

Using command

r = roots(poly(A))

displays (in format short)

```
r =

    4.0000
   -3.0000
    1.0000
```

Note: If exact arithmetic had been used, then the roots of the characteristic polynomial for this matrix would have been integers 4, −3, and 1. Displaying **r** in format long e shows that a small amount of roundoff error occurred in the computation and hence MATLAB could not display the exact integer values. Such situations frequently occur in finding the roots of a characteristic polynomial in MATLAB (and other software).

The eigenvalues of A are $\lambda = 4, -3,$ and 1.

Once we have the eigenvalues λ of a matrix A, the eigenvectors are determined as nontrivial solutions x of the homogeneous system $(A-\lambda I)x = 0$. To find $x \neq 0$, compute rref($A-\lambda I$) and construct the general solution of the homogeneous system. Linearly independent eigenvectors corresponding to λ are often obtained by extracting a basis for the general solution. This is equivalent to finding a basis for the null space of $A-\lambda I$ or a basis for the kernel of the linear transformation defined by $L(x) = (A-\lambda I)x$. It is also a fact that eigenvectors corresponding to different eigenvalues are linearly independent. (An alternate approach uses routine **homsoln** applied to $A - \lambda I$ for each eigenvalue λ, which must be exactly specified.)

Example 2. For $A = \begin{bmatrix} 5 & -8 & -1 \\ 4 & -7 & -4 \\ 0 & 0 & 4 \end{bmatrix}$, as defined in Example 1, the eigenvalues are $\lambda = 4, -3,$ and 1. To find the corresponding eigenvectors in MATLAB proceed as follows.

Case $\lambda = 4$: MATLAB command

M = rref(A − 4*eye(size(A)))

displays

```
M =

     1     0    -1
     0     1     0
     0     0     0
```

The general solution of $(\boldsymbol{A} - 4\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ is given by

$$x_3 = r, \qquad x_2 = 0, \qquad x_1 = r.$$

Hence $\boldsymbol{x} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and we take $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to eigenvalue $\lambda = 4$. Note that we could have set constant r to any nonzero value to obtain an eigenvector. Hence eigenvectors corresponding to an eigenvalue are not unique.

Case $\lambda = -3$: MATLAB command

M = rref(A − (−3)*eye(size(A)))

displays

```
M =

     1    -1     0
     0     0     1
     0     0     0
```

The general solution of $(\boldsymbol{A} + 3\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$ is given by

$$x_3 = 0, \qquad x_2 = r, \qquad x_1 = r$$

Hence $\boldsymbol{x} = r \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and we take $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ as an eigenvector corresponding to eigenvalue $\lambda = -3$.

Case $\lambda = 1$: MATLAB command

$$\mathbf{M = rref(A - 1{*}eye(size(A)))}$$

displays

```
M =

     1    -2     0
     0     0     1
     0     0     0
```

The general solution of $(\boldsymbol{A} - 1\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$ is given by

$$x_3 = 0, \qquad x_2 = r, \qquad x_1 = 2r.$$

Hence $\boldsymbol{x} = r\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and we take $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ as an eigenvector corresponding to eigenvalue $\lambda = 1$.

Since matrix $\boldsymbol{A}$ had 3 distinct eigenvalues it follows that the eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent.

Example 3. Let $\boldsymbol{A} = \begin{bmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ -4 & 4 & 3 \end{bmatrix}$. Command **r = roots(poly(A))** reveals that the eigenvalues of $\boldsymbol{A}$ are $\lambda = 3, 3, -1$. We find the eigenvectors corresponding to $\lambda = 3$ as follows. (We omit the case for $\lambda = -1$.)

Case $\lambda = 3$:

$$\mathbf{M = rref(A - 3{*}eye(size(A)))}$$

displays

```
M =

     1    -1     0
     0     0     0
     0     0     0
```

Hence $x_3 = r, x_2 = s, x_1 = s$ and we have

$$\boldsymbol{x} = \begin{bmatrix} s \\ s \\ r \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that both $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to $\lambda = 3$. Since r and s are arbitrary it follows that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are a pair of linearly independent eigenvectors corresponding to $\lambda = 3$. (Alternatively, **homsoln(A - 3*eye(size(A)))** produces the same eigenvectors.)

Warning: If a matrix has a k-times repeated eigenvalue, then it is possible that there will be fewer than k corresponding linearly independent eigenvectors. Such matrices are called **defective**. In a computing environment defective matrices may be difficult to recognize because of roundoff error within computations.

In MATLAB type

help eig

The display gives a description of the **eig** command. We are concerned only with the following features:

- **eig(A)** displays a vector containing the eigenvalues of square matrix $\boldsymbol{A}$.
- **[v,d] = eig(A)** displays the eigenvectors of $\boldsymbol{A}$ as columns of matrix $\boldsymbol{v}$ and the diagonal matrix $\boldsymbol{d}$ contains the corresponding eigenvalues.

We illustrate command **eig** in the following example.

Example 4. Enter matrix $\boldsymbol{A} = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 2 & 1.5 \\ -5 & 0 & .5 \end{bmatrix}$ into MATLAB . Command

r = eig(A)

displays

```
r =

    2.0000
    0.5000
    3.0000
```

Command

$$[\mathbf{v},\mathbf{d}] = \mathbf{eig(A)}$$

displays

```
v =                                    d =
        0         0    0.4082              2.0000        0         0
   1.0000    -.7071    0.4082                   0   0.5000         0
        0     .7071   -0.8165                   0        0    3.0000
```

The columns of $\boldsymbol{v}$ are the eigenvectors of $\boldsymbol{A}$ corresponding to the eigenvalues in the diagonal entries of the same numbered column of $\boldsymbol{d}$. It is MATLAB 's convention that the eigenvectors are scaled (multiplied by a nonzero scalar) so that their norm is 1. Had we done the computations by hand the matrix $\boldsymbol{v}$ could have been displayed as

```
0   0   1
1  -1   1
0   1  -2
```

Warning: MATLAB computes eigenvalues and eigenvectors by methods different from those we have studied. The results are quite accurate, but may appear different from the corresponding hand calculations.

Exercises 13.2

1. Use commands **poly** and **roots** to find the characteristic polynomial and the eigenvalues of each of the following matrices. Record your results below each matrix.

a) $A = \begin{bmatrix} 4 & -2 & -5 \\ 1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$ **b)** $B = \begin{bmatrix} -6 & 8 & 1 \\ -4 & 6 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

________________ ________________

________________ ________________

c) $C = \begin{bmatrix} -1/2 & 1 & -1/2 \\ -1/2 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$

d) $D = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

________________ ________________

________________ ________________

2. Here we investigate the eigenvalues of upper triangular matrices. Perform the following experiments. Record the matrix $\boldsymbol{A}$ and its eigenvalues. Look for a connection between the entries of $\boldsymbol{A}$ and its eigenvalues.

n = 3; A = triu(fix(10*rand(n))), r = roots(poly(A))

Repeat the experiment several times. Then change n to 4 and do the experiment. Complete the following conjecture:

The eigenvalues of an upper triangular matrix are ________________.

Check your conjecture by changing n to 5 and repeat the experiment several times.

3. In the MATLAB commands in Exercise 2, replace **triu** by **tril** and investigate the eigenvalues of a lower triangular matrix. Look for a connection between the entries of $\boldsymbol{A}$ and its eigenvalues. Complete the following conjecture:

The eigenvalues of a lower triangular matrix are ________________.

4. Using the results from Exercises 2 and 3 fill in the following conjecture.

The eigenvalues of a diagonal matrix are ________________.

Give a reason for this conjecture based on ideas relating triangular and diagonal matrices.

____

5. Find the eigenvalues and eigenvectors for each of the following. Record your results below each matrix.

a) $A = \begin{bmatrix} 3 & 0 & -1 \\ -1 & -6 & 9 \\ -1 & 0 & 3 \end{bmatrix}$ b) $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ -2 & -2 & -1 \end{bmatrix}$

6. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Find the eigenvalues of A. (The eigenvalues will be complex even though the matrix A is real.) Find the eigenvectors as in Example 2. Record your results below.

7. Find the eigenvalues and eigenvectors of complex matrix $A = \begin{bmatrix} 3-2i & -1-2i & 0 \\ 0 & 4 & 0 \\ -2+6i & 5i & 2+i \end{bmatrix}$ using the methods in Examples 1 and 2. Record your results below. (Note that A has complex eigenvalues, but its eigenvectors are real.)

8. Use **eig** on the matrices in Exercise 1. Check the eigenvalues with those you computed using **poly** and **roots**.

9. Use **eig** on the matrices in Exercise 5. Check your results from Exercise 5.

10. Do Exercise 7 using **eig**.

Section 13.3

Further Experiments and Applications

Following are a collection of experiments on eigenvalues and eigenvectors that provide opportunities to study properties of these concepts. We also provide an introduction to diagonalizable matrices using MATLAB experiments. Applications involving powers of a matrix, Markov matrices, and graph theory are explored briefly.

Exercises 13.3

1. Let $\boldsymbol{A} = \begin{bmatrix} -11 & 0 & -1 & 13 \\ 12 & 1 & 1 & -13 \\ 10 & 0 & 4 & -10 \\ -5 & 0 & -1 & 7 \end{bmatrix}$. Compute the eigenvalues of $\boldsymbol{A}$, $\boldsymbol{A}^2$, and $\boldsymbol{A}^3$. Record your results below.

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