Cardinal-Utility Matching Markets: The Quest for Envy-Freeness, Pareto-Optimality, and Efficient Computability

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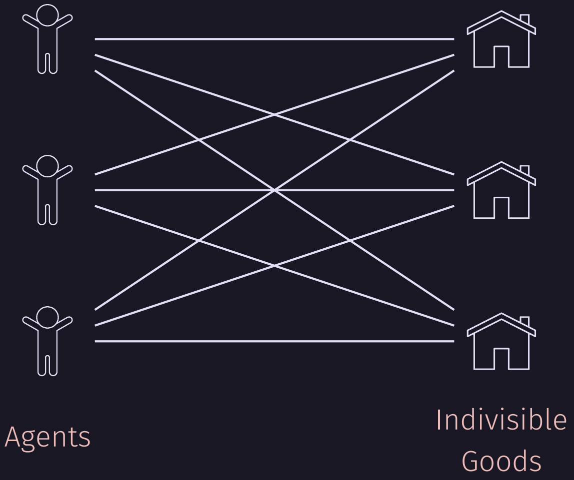
Problem Setting



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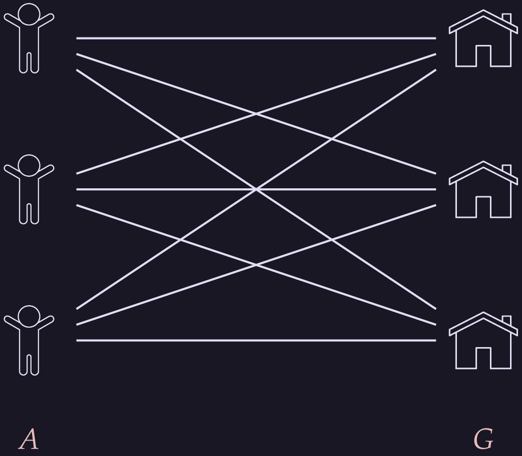

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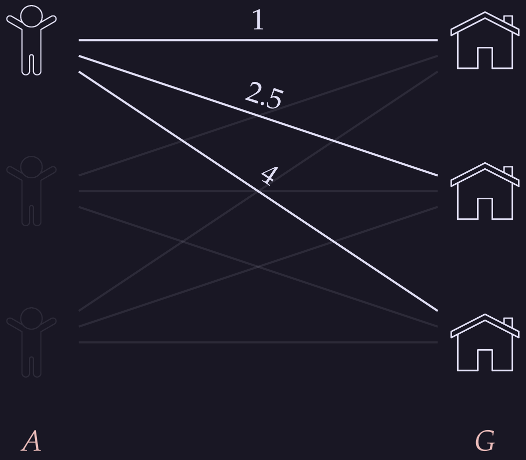

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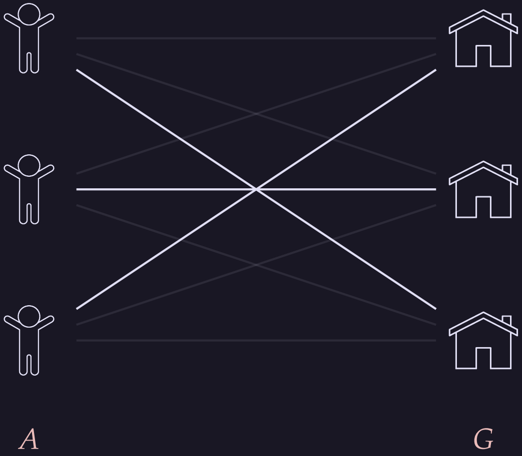

Formal Setup

We are given:

- agents A,
- goods G,
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Task

Find perfect matching satisfying desirable properties (fairness, efficiency, etc.).

Question

Why cardinal utilities instead of ordinal?

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Theorem (Immorlica et al. 2017)

Cardinal-utility mechanisms can improve the utility of all agents by a $\theta(\log(n))$-factor over ordinal mechanisms.

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3. Find market equilibrium in the resulting one-sided, linear matching market
4. Run lottery based on Birkhoff-von-Neumann theorem

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1. x is a fractional perfect matching.
2. No agent overspends, i.e. $p \cdot x_i \leq 1$.
3. Every agent gets optimum bundle, i.e. $u_i \cdot x_i = \max\{u_i \cdot y \mid \sum_{j \in G} y_j = 1, p \cdot y \leq 1\}$.

Hylland-Zeckhauser Mechanism III

Theorem (Hylland, Zeckhauser 1979)

An HZ equilibrium always exists. If (x, p) is an HZ equilibrium, then x is Pareto-optimal and envy-free.

Theorem (He et al. 2018)

The HZ mechanism is incentive-compatible in the large.

But Wait...

Question

But... how do we actually find an HZ equilibrium?

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Theorem (Chen, Chen, Peng, Yannakakis 2022)

The problem of computing an ϵ-approximate HZ-equilibrium is PPAD-hard when $\epsilon = 1/n^c$ for any constant $c > 0$.

Also, challenging in practice!

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Can we find an envy-free (EF) and Pareto-optimal (PO) allocation in polynomial time?

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Theorem (Tröbst, Vazirani 2024)

There is a polynomial time mechanism which is $(2+\epsilon)$-EF, $(2+\epsilon)$-IC and PO.

PPAD-Hardness

Proof Strategy

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There is a polynomial reduction from $\frac{3}{n}$-approximate HZ to finding EF+PO allocations.

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EF+PO and HZ are quite different:

1. HZ may have only irrational solutions, but there are always rational EF+PO solutions
2. HZ little structure (fixed point), but EF+PO is polyhedral

Strategy:

1. Modify the instance in a clever way

Proof Strategy II

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1. Modify the instance in a clever way
2. Use the second welfare theorem: get prices and budgets from Pareto-optimality.
3. Main idea: use envy-freeness and linearity to show that budgets must be (approximately) equal.

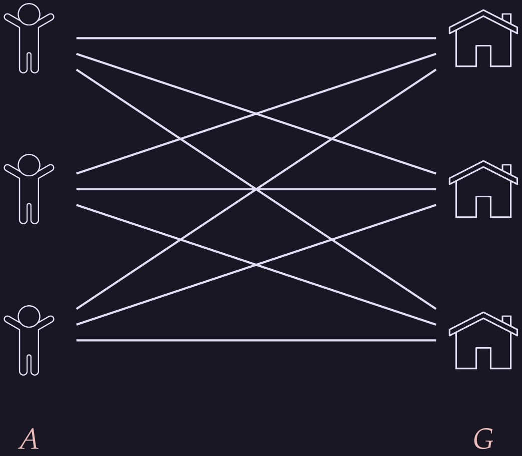
Lemma (Optimal Bundles)

We can find budgets b and prices p, so that for every agent i, x_i is an optimum solution to

$$\begin{aligned} \max \quad & u_i \cdot x_i \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1, \\ & p \cdot x_i \leq b_i, \\ & x_i \geq 0. \end{aligned}$$

$\approx$ Second Welfare Theorem, get prices by setting up correct primal and dual LPs

Idea 1: Expand the Instance ($k = 4$)



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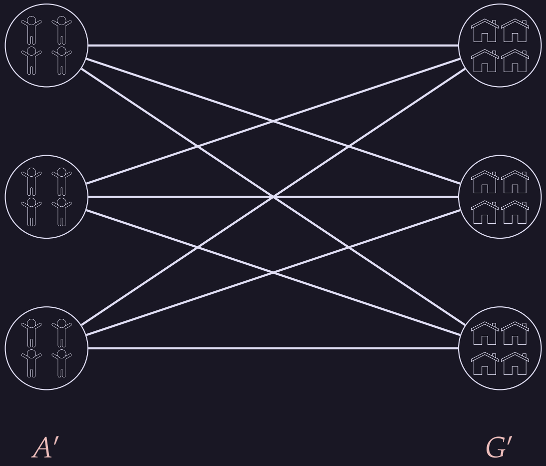

After modifying the instance:

Lemma

Let $i, i' \in A$ be such that utilities agree up to one good where they differ by at most ϵ. Then $|b_i - b_{i'}| \leq 5n^2\epsilon$.

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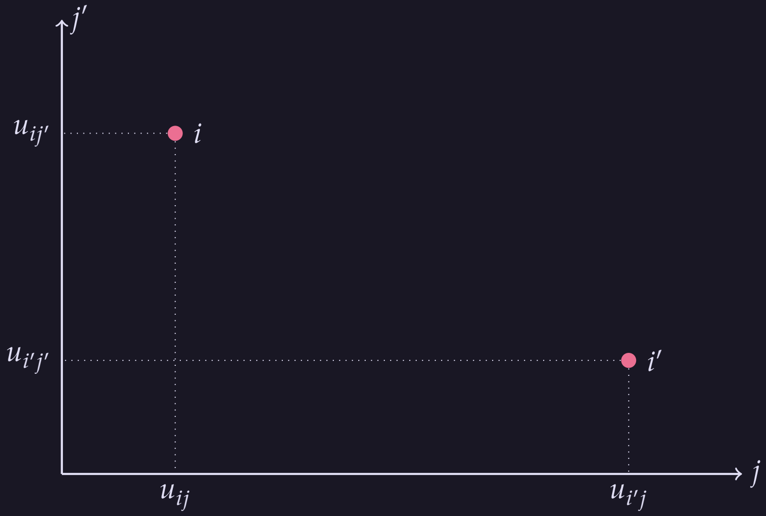
Let $i, i' \in A$ be such that utilities agree up to one good where they differ by at most ϵ. Then $|b_i - b_{i'}| \leq 5n^2\epsilon$.

Proof. Suppose $b_i > b_{i'}$. Then i gets a better bundle than i' due to non-satiation. i' agrees that i's bundle is better: envy! □

Idea 3: Interpolation



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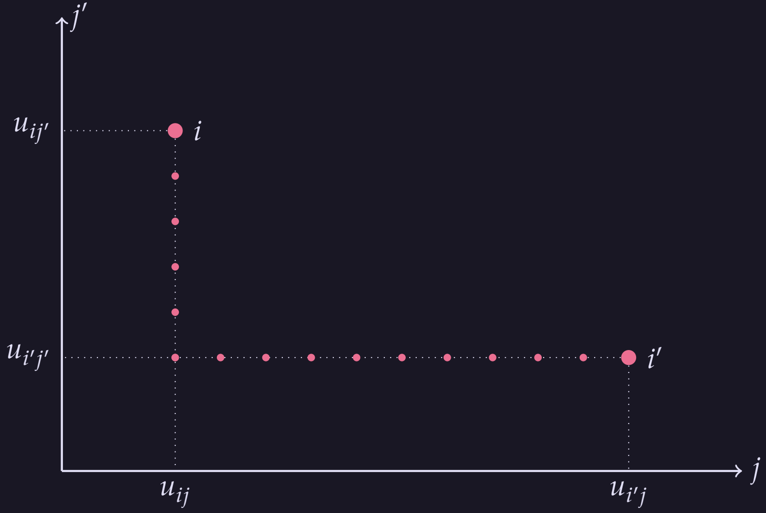

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So $|b_i - b_{i'}| \leq 5n^3$.

Completely useless! ☹

Generalizing to Optimal Bundle Equality

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Let $i, i' \in A$ such that i and i' agree on which bundles are optimal bundles. Then $b_i = b_{i'}$.

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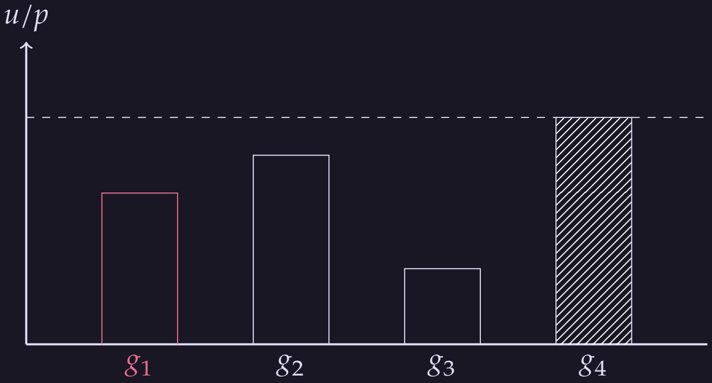
Caveat:

- In HZ, optimum bundles depend on utilities, prices, and the budget of the agent.
- For the lemma, agents must agree on the optimum bundles at all possible budgets.

Key Idea: Optimal Bundles Rarely Change



Without matching constraint: bundles only change when critical bang per buck treshhold is reached.

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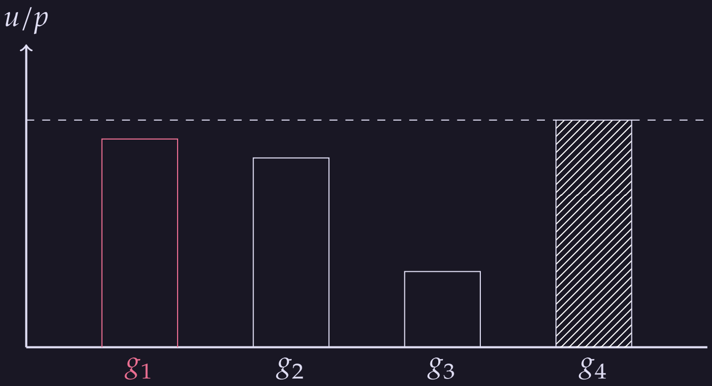

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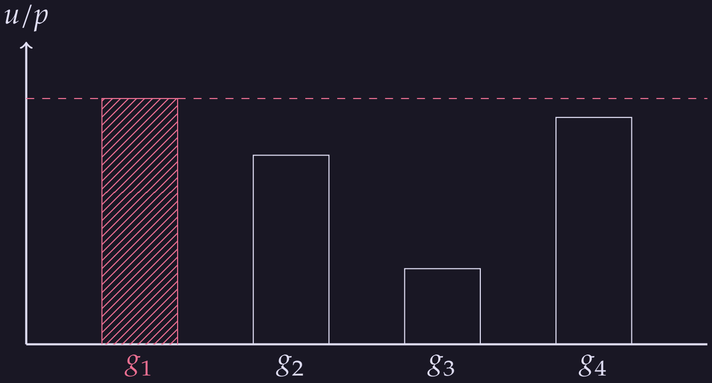

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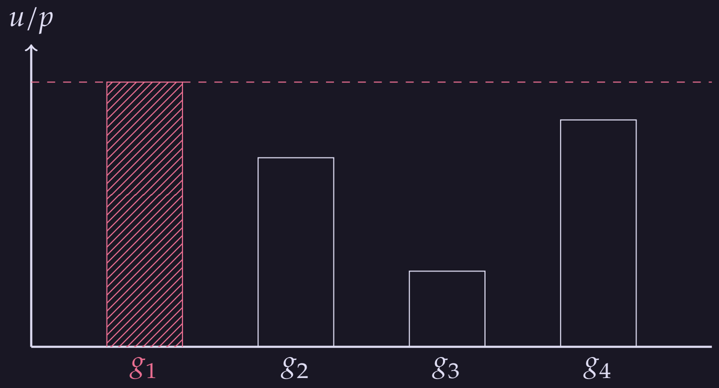

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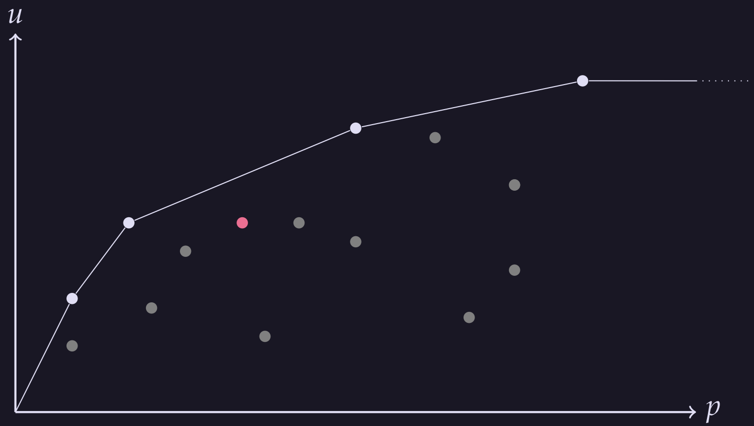
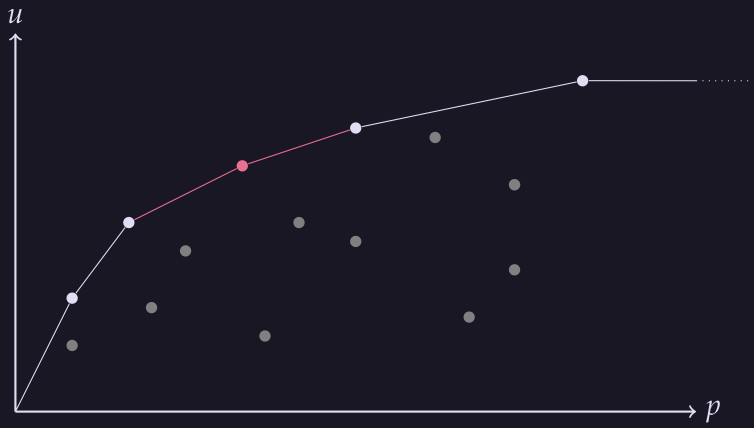
Optimal Bundles in HZ



With matching constraint: more complex characterization of optimal bundles.

Optimal Bundles in HZ



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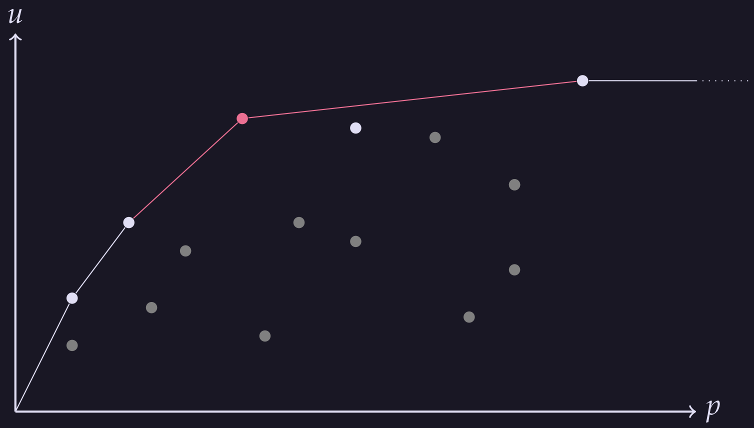

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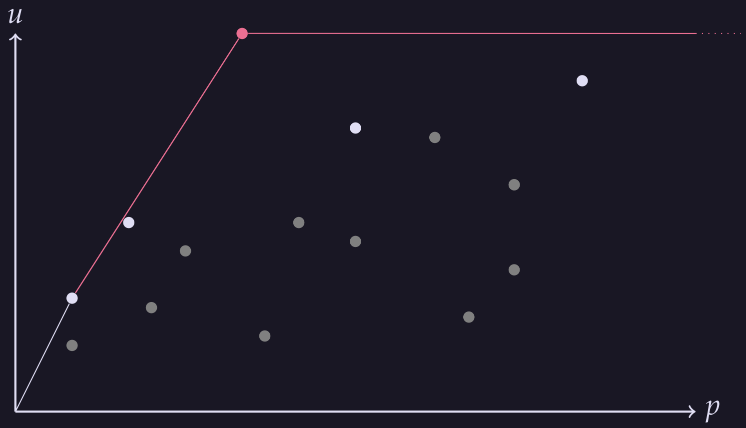

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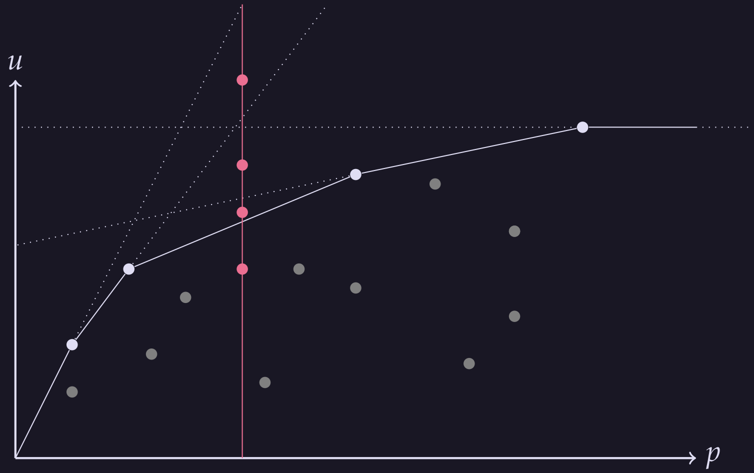

With matching constraint: more complex characterization of optimal bundles.

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Proof. Between two agents, at most $2n^2$ changes can happen. Each contributes at most $5\epsilon n^2$. □

Bringing It Together II

Theorem

If $\epsilon \leq \frac{1}{5n^5}$ and $k = \frac{n^3}{\epsilon}$, then (x, p) is a $\frac{3}{n}$-approximate HZ equilibrium in the original instance.

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Theorem

If $\epsilon \leq \frac{1}{5n^5}$ and $k = \frac{n^3}{\epsilon}$, then (x, p) is a $\frac{3}{n}$-approximate HZ equilibrium in the original instance.

Theorem

The problem of finding an EF+PO allocation in one-sided cardinal-utility matching market is PPAD-hard.

Nash Bargaining

Nash Bargaining Convex Program

Hosseini, Vazirani 2021: Let's use Nash bargaining instead:

$$\begin{aligned} \max_{x} \quad & \sum_{i \in A} \log(u_i(x)) \\ \text{s.t.} \quad & \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\ & \sum_{j \in A} x_{ij} \leq 1 \quad \forall i \in A, \\ & x \geq 0. \end{aligned}$$

Concrete polynomial time algorithms given in Panageas, Tröbst, Vazirani 2022.

Question

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2. Polynomial time computability (by convex program)

⇒ What else?

Theorem (Tröbst, Vazirani 2024)

If x is a Nash bargaining solution, then x is 2-approximately envy free.

Properties of Nash Bargaining II

Theorem (Tröbst, Vazirani 2024)

If x is a Nash bargaining solution, then x is 2-approximately envy free.

Theorem (Tröbst, Vazirani 2024)

The Nash-bargaining-based mechanism is 2-approximately incentive compatible.

Nash bargaining is a practical HZ alternative for one-sided cardinal-utility matching markets.

Thank you!