The Intractibility of Hylland-Zeckhauser and its Aftermath

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Theory Seminar, October 13, 2023

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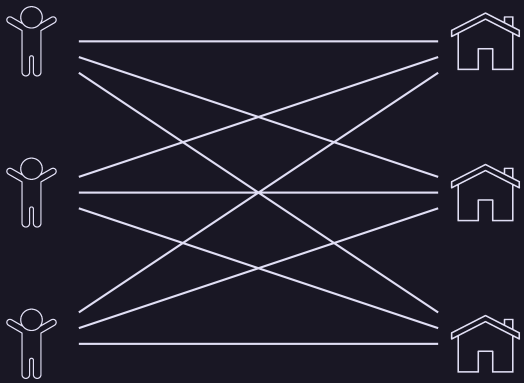
The Hylland-Zeckhauser Scheme

Problem Setting



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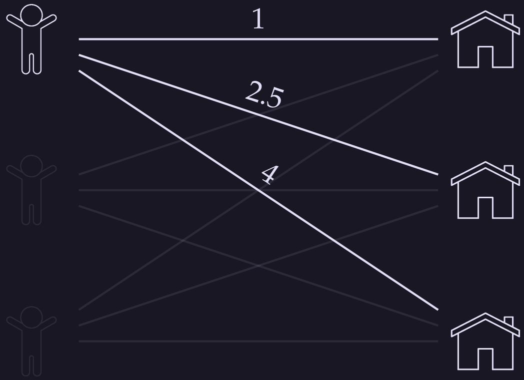

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4. Run a lottery based on the equilibrium allocation using the Birkhoff-von-Neumann theorem.

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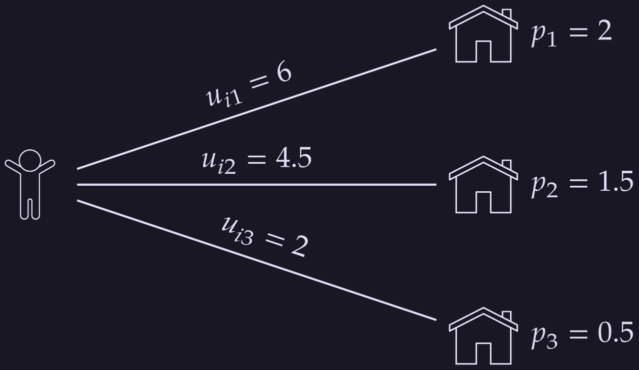
- each agent gets a cheapest optimal bundle.

Cheapest Optimal Bundles

In HZ, agents get **utility-maximizing bundles** of goods at market prices. If there are multiple optimal bundles, pick a **cheapest** one.



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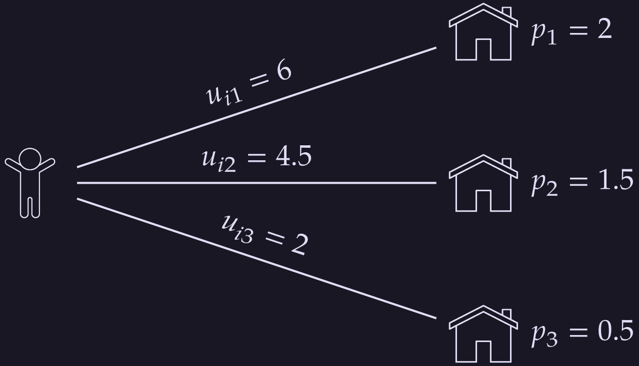

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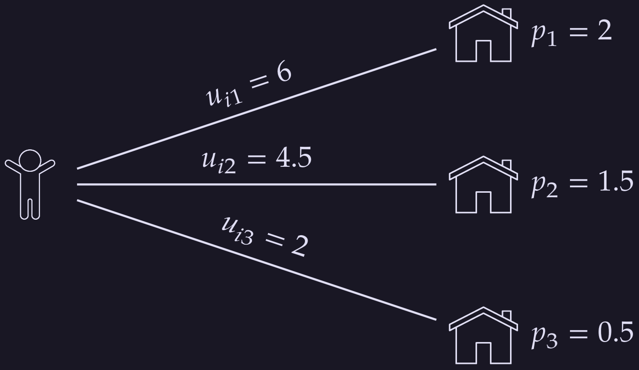

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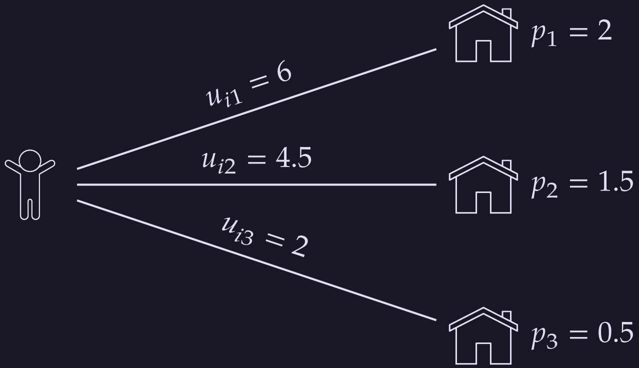

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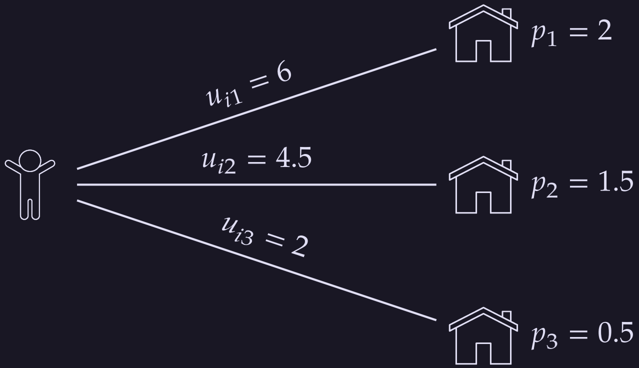

$$x_{i1} = 0, x_{i2} = 0.5, x_{i3} = 0.5 \Rightarrow \mathbf{u_i} = \mathbf{3.25}$$

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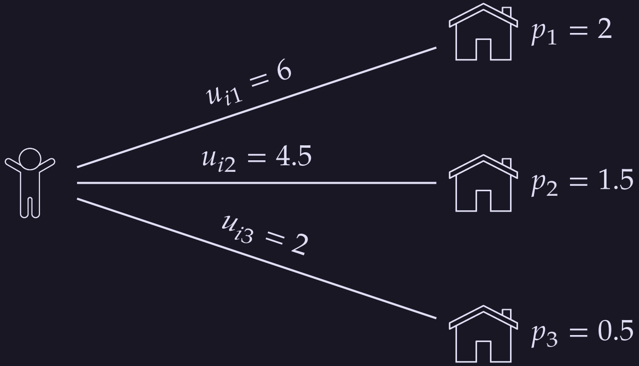

$$x_{i1} = 0.\overline{3}, x_{i2} = 0, x_{i3} = 0.\overline{6} \Rightarrow \mathbf{u_i = 3.\overline{3}}$$

Properties of the Hylland-Zeckhauser Scheme

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- fair in the sense of envy-freeness,
- efficient in the sense of Pareto-optimality, and
- strategy-proof in the sense of incentive compatibility in the large.

Let x be some allocation (i.e. fractional perfect matching).

Definition

Envy-Free For agents i, i' - we say i envies i' if

$$\sum_{j \in G} u_{ij} x_{ij} < \sum_{j \in G} u_{ij} x_{i'j}.$$

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In other words: no agent thinks that another agent got a better bundle than they did.

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In other words: there is no way to improve one agent without making another agent worse off.

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The fair division community has largely moved on to other settings. But this problem is far from solved!

Intractibility and Impossibility Results

The HZ scheme depends on the ability to compute an HZ equilibrium. So what is the state of the art?

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- Polynomial time algorithm for bi-valued utilities (Vazirani, Yannakakis 2020).

Computation of Market Equilibria

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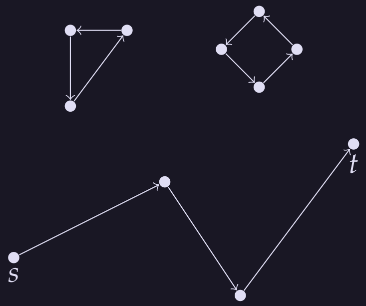
- Finding an HZ equilibrium is in the complexity class FIXP.
- Finding an approximate HZ equilibrium is in the complexity class PPAD.

Problems in the class PPAD (Polynomial Parity Argument on Digraphs) can be reduced to a kind of path-following problem in an exponentially large directed graph:



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From a computational perspective, the problem that Hylland and Zeckhauser solved in 1979 is once again open!

Conclusion

"In my opinion, if the theorem that Nash equilibria exist is considered relevant to debates about (say) free markets versus government intervention, then the theorem that finding those equilibria is PPAD-complete should be considered relevant also."

– Scott Aaronson (Why Philosophers Should Care About Computational Complexity)

A Path Forward

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- Strategy-proofness (ideally DSIC)

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- Fairness + efficiency: ???

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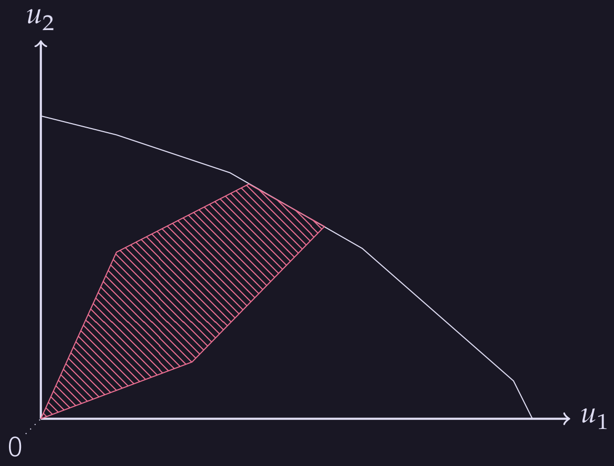
Compared to HZ:

- HZ-equilibria can be irrational and
- the best-known algorithm to find them uses algebraic cell decomposition which takes at least $\omega(n^{5n^2})$ time.

Pareto-Optimal and Envy-Free Solutions



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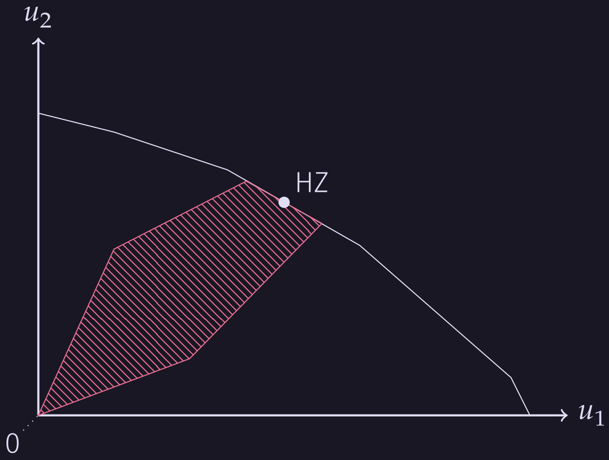

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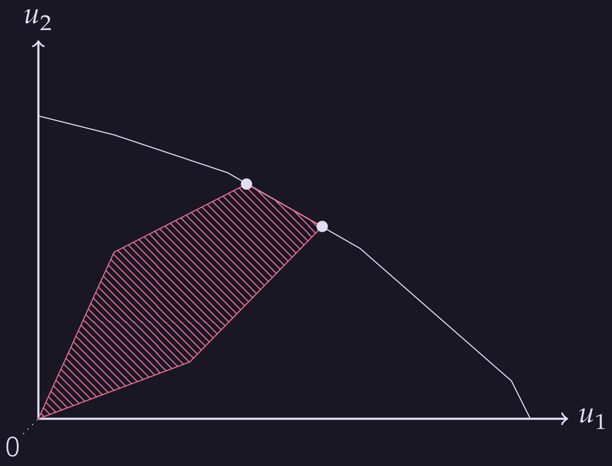

Nash Bargaining

Nash Bargaining Point

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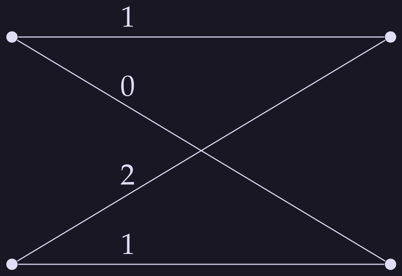
Definition

Let U be the set of utility vectors achievable by fractional matchings. The Nash bargaining point is

$$\arg\max_{u \in U} \prod_{i \in A} u_i.$$

Fairness of Nash Bargaining

Nash bargaining points have nice properties in general such as Pareto-optimality. But what about fairness?



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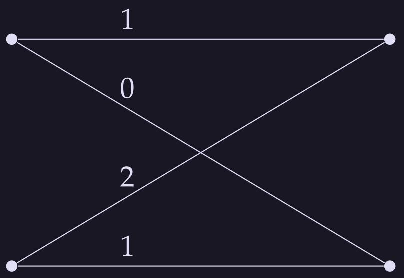

Not envy free! This also shows that Nash bargaining is not incentive compatible!

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- proportionally fair, i.e. increasing one agent's utility by $2x$ must reduce other agents utilities by $0.5x$, and
- $\frac{1}{2}$-equal-share fair, i.e. every agent gets at least half of their average utility (Panageas, Tröbst, Vazirani 2022).

Nash Bargaining Convex Program

Big advantage: Nash bargaining is a convex program!

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\begin{aligned}
\max_{x} \quad & \sum_{i \in A} \log(u_i(x)) \\
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So we can compute this solution in polynomial time!

Theorem (Panageas, Tröbst, Vazirani 2022)

We can compute an ϵ-approximate Nash bargaining solution after $O\left(\frac{n \log n}{\epsilon^2}\right)$ iterations of a multiplicative-weights type algorithm. Each iteration can be carried out in $O(n^2)$ time.

Theorem (Panageas, Tröbst, Vazirani 2022)

We can compute an ϵ-approximate Nash bargaining solution after $O\left(\frac{n^3 \kappa^2}{\epsilon}\right)$ iterations of a conditional gradient type algorithm. Each iteration consists of computing a max-weight bipartite matching ($O(n^3)$ time).

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- Two-sided matching markets
  - There are extensions of HZ (see Echenique, Miralles, Zhang 2020) but they do not have desirable properties.
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  - HZ does not exist (Hylland, Zeckhauser 1979), even under strong assumptions (Garg, Tröbst, Vazirani 2022).

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- More general utilities

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Future Work

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- rational envy-free and Pareto-optimal solutions which are easier to compute but still exponential time, and
- Nash bargaining, which is easy to compute and efficient but has much weaker fairness properties.

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- Are there natural dynamics (ala tatonnement) which converge to the Nash bargaining point?
- What can be said about extensions where HZ does not exist?

Thank your for listening!