Cardinal-Utility Matching Markets: The Quest for Envy-Freeness, Pareto-Optimality, and Efficient Computability

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Oberseminar Diskrete Optimierung
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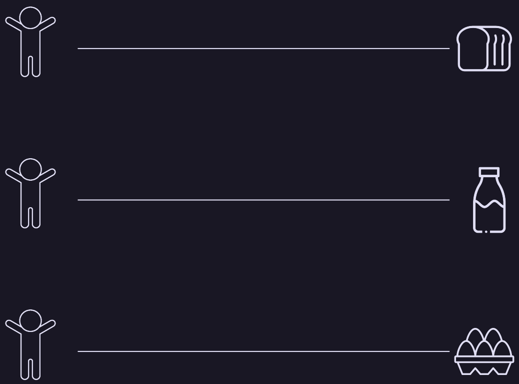
Introduction

Agents

Goods

Introduction




Agents
Goods

Ordinal vs Cardinal

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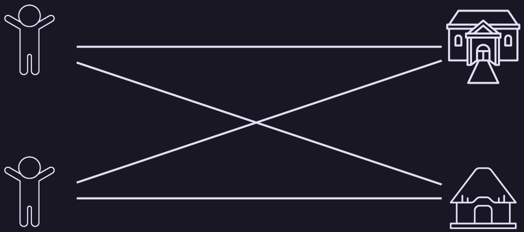
Fairness and Lotteries

Cannot achieve fairness without lotteries:



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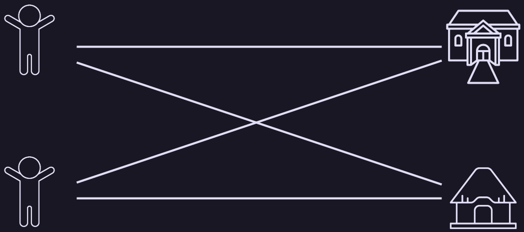


Only fair allocation: run a 50/50 lottery!

Birkhoff-von-Neumann Theorem

Theorem (Birkhoff 1946, von-Neumann 1953)

Let $(x_{ij})_{i \in A, j \in G}$ be a fractional perfect matching. Then x can be decomposed into a convex combination of n^2 integral perfect matchings. This can be done in polynomial time.

$\Rightarrow$ allows us to treat any fractional perfect matching as a lottery.

Definition of Fairness

Given allocation $(x_{ij})_{i \in A, j \in G}$ and utilities $(u_{ij})_{i \in A, j \in G}$:

Definition (Envy)

Agent i envies agent i' if

$$u_i \cdot x_i < u_i \cdot x_{i'}$$

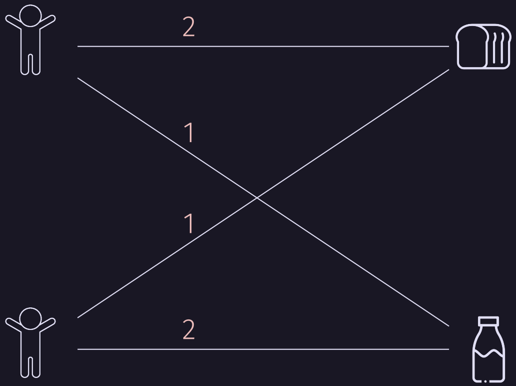
i.e.

$$\sum_{j \in G} u_{ij} x_{ij} < \sum_{j \in G} u_{ij} x_{i'j}.$$

Definition (Envy-freeness)

Allocation x is envy-free if no agent envies another.

Efficiency



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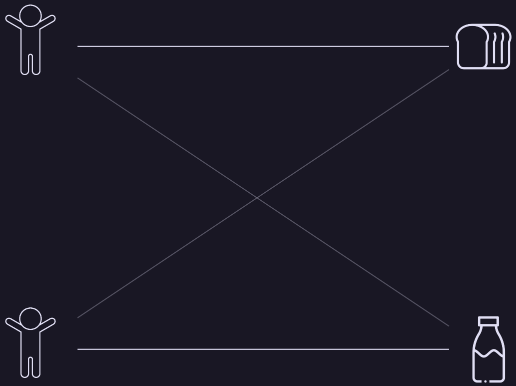

Given allocation $(x_{ij})_{i \in A, j \in G}$ and utilities $(u_{ij})_{i \in A, j \in G}$:

Definition (Pareto-Better)

Let $(y_{ij})_{i \in A, j \in G}$ be another allocation. y is Pareto-better than x if $u_i \cdot y_i \geq u_i \cdot x_i$ for all i and $u_i \cdot y_i > u_i \cdot x_i$ for some i.

Definition (Pareto-Optimality)

x is Pareto-optimal if there is no Pareto-better allocation.

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Note: can't do better than $\theta(\log n)$-PO with only ordinal information (Immorlica et al.)

Question

Is there always a fractional perfect matching which is fair (envy-free) and efficient (Pareto-optimal)?

If so, can we find it in polynomial time?

In the rest of the talk:

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1. Yes, there is always an EF+PO allocation.

Overview

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2. No, we can't compute it in polynomial time.

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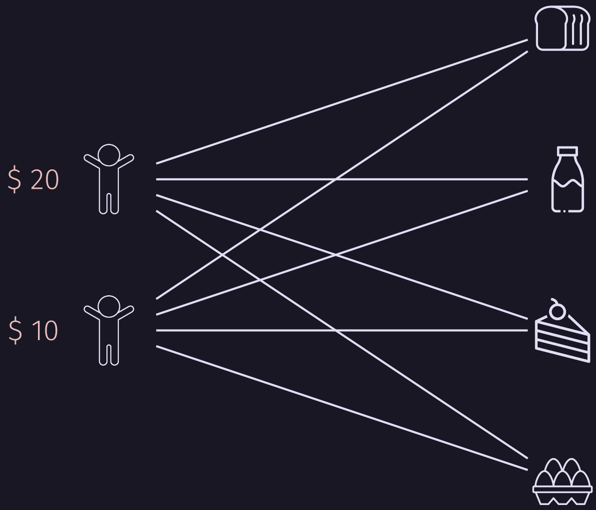
1. Yes, there is always an EF+PO allocation.
2. No, we can't compute it in polynomial time.
3. But we can get a 2-approximation.

Hylland Zeckhauser Mechanism

Fisher Market



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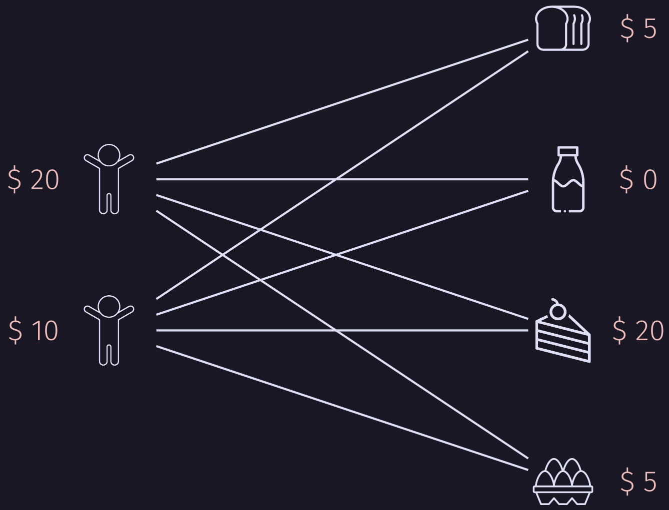

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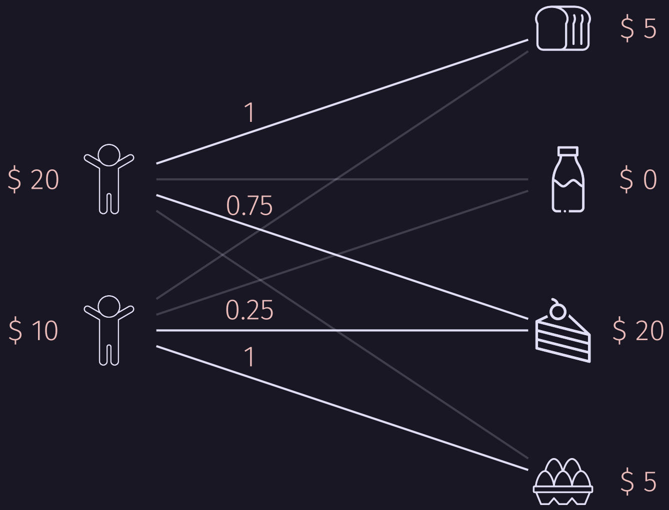

Linear Fisher Market Model

Given

- agents A,
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Definition (Fisher Market Equilibrium)

A Fisher market equilibrium consists of an allocation $(x_{ij})_{i \in A, j \in G}$ and non-negative prices $(p_j)_{j \in G}$ such that

- every agent spends their budget on a utility-maximizing bundle,
- the market clears.

Hylland Zeckhauser Mechanism

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4. Turn equilibrium allocation into lottery via Birkhoff-von-Neumann theorem

⇒ Intuitively: HZ ≈ Fisher market + matching + rounding

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1. x is a fractional perfect matching.
2. No agent overspends, i.e. $p \cdot x_i \leq 1$.
3. Every agent gets optimum bundle, i.e. $u_i \cdot x_i = \max\{u_i \cdot y \mid \sum_{j \in G} y_j = 1, p \cdot y \leq 1\}$.

Envy-Freeness of HZ

Theorem (Hylland, Zeckhauser 1979)

An HZ equilibrium is always envy-free.

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Proof. Consider agents i and i'. $x_{i'}$ is a feasible bundle for agent i. But x_i is an optimal bundle for agent i. So $u_i \cdot x_i \geq u_i \cdot x_{i'}$. □

Pareto-Optimality of HZ

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PARETO-OPTIMALITY OF HZ

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Proof. Let y be another allocation, assume it is strongly Pareto-better. For every agent i, $u_i \cdot y_i > u_i \cdot x_i$.

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But x_i was optimal, so $p \cdot y_i > 1$. Sum over i to get:

$$n < \sum_{i \in A} p \cdot y_i = \sum_{i \in A} \sum_{j \in G} p_j y_{ij} = \sum_{j \in G} p_j \sum_{i \in A} y_{ij} \leq \sum_{j \in G} p_j \leq n.$$

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Contradiction! □

Existence

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HZ equilibria always exist.

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Proof.

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Proof. Complicated… (based on Kakutani's fixed point theorem, similar to Brouwer's fixed point theorem)

How do you find an HZ equilibrium?

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Theorem (Devanur, Papadimitrious, Saberi, Vazirani 2002)

Can find Fisher market equilibria in polynomial time using combinatorial, flow-based algorithm. Always finds rational equilibrium.

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Conjecture

HZ algorithm = DPSV + matching? Should be doable!

Theorem (Alaei, Khalilabadi, Tardos 2017)

There is an algorithm based on algebraic cell decomposition which checks $> n^{5n^2}$ cells.

Theorem (Vazirani, Yannakakis 2020)

There is a polynomial time algorithm for $\{0, 1\}$ utilities.

⇒ galactic running time or restrictive utilities...

Irrationality

Theorem (Vazirani, Yannakakis 2020)

There are instances of HZ in which there is a unique equilibrium with irrational allocations and prices!

$\Rightarrow$ rules out exact, combinatorial algorithm

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Theorem (Vazirani, Yannakakis 2020)

HZ is in FIXP, approximate HZ is in PPAD.

Theorem (Chen, Chen, Peng, Yannakakis 2022)

The problem of computing an ϵ-approximate HZ-equilibrium is PPAD-hard for $\epsilon = 1/n^c$ for any constant $c > 0$.

Hardness

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The problem of computing an ϵ-approximate HZ-equilibrium is PPAD-hard for $\epsilon = 1/n^c$ for any constant $c > 0$.

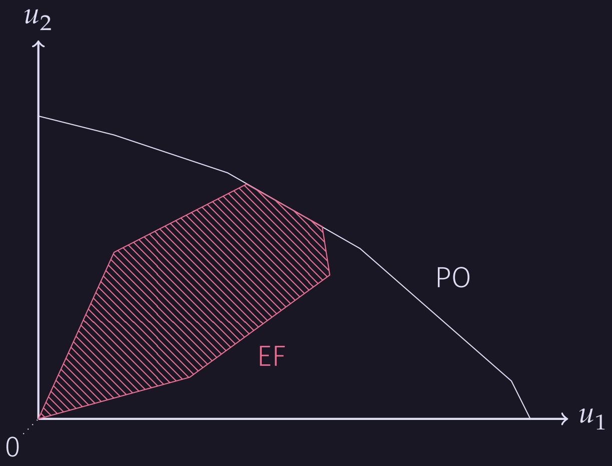
$\Rightarrow$ computing HZ-equilibria is as hard as

- computing general Nash-equilibria,
- computational versions of Kakutani's / Brouwer's fixed-point theorems.

☹

EF+PO and HZ

Rationality of EF+PO



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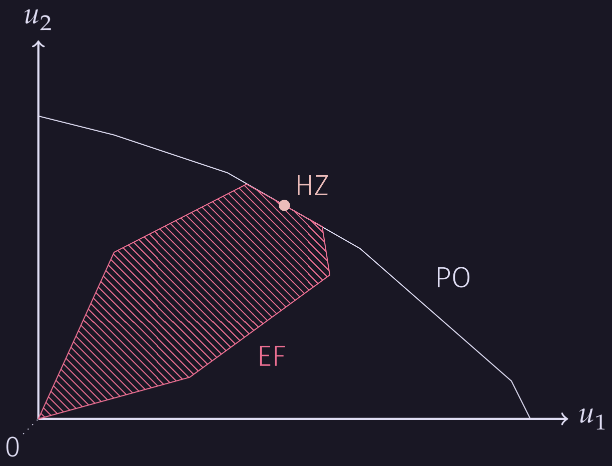

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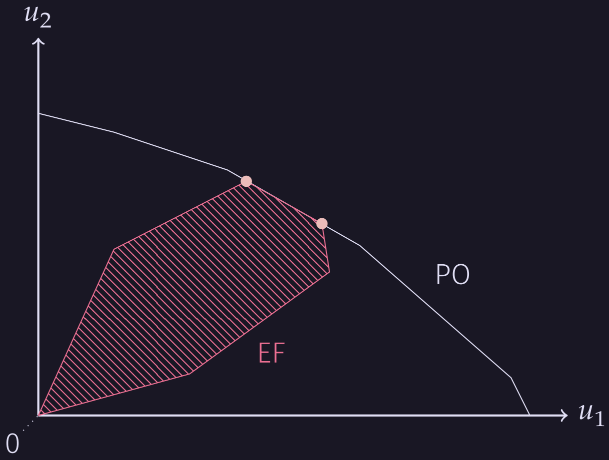

Main Result

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Theorem (Tröbst, Vazirani 2024)

Finding an EF+PO allocation is PPAD-hard.

Strategy: polynomial reduction of approximate HZ to EF+PO

HZ instance

Proof Strategy

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HZ instance $\Longrightarrow$ mod. instance

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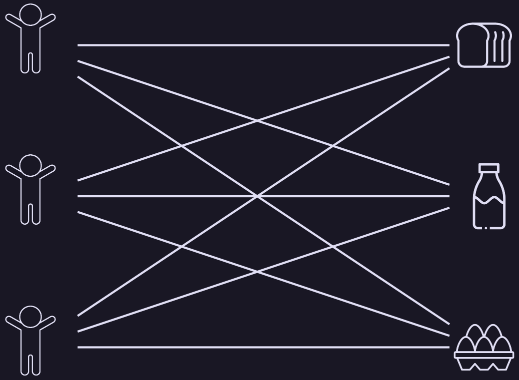
apx. HZ $\Longleftarrow$ EF+PO

From EF+PO to HZ



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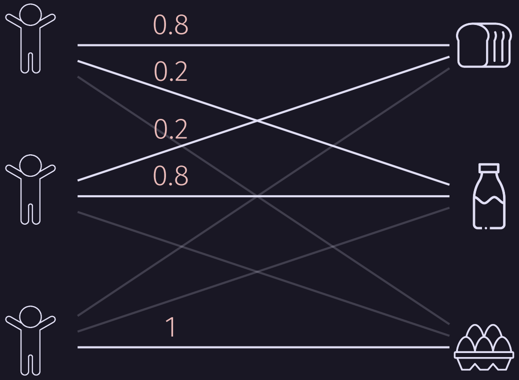
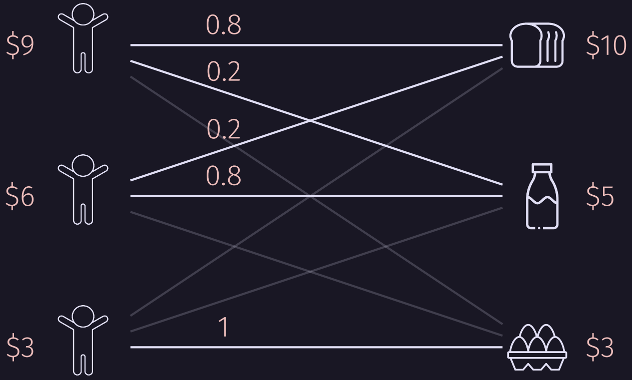
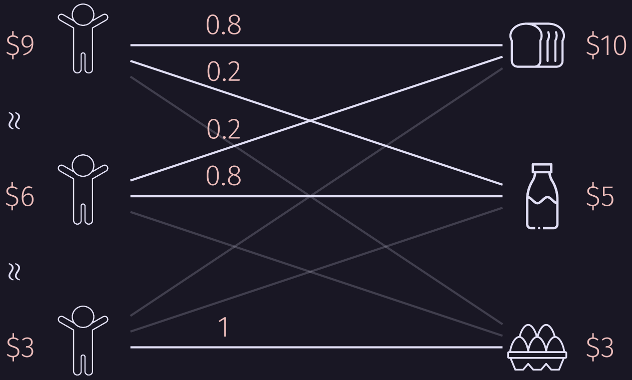




Step 1: conjure up prices

From EF+PO to HZ



Step 2: budgets apx. equal

Strategy:

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Proof Strategy Review

Strategy:

1. Modify the instance in a clever way
2. Use the second welfare theorem: get prices and budgets from Pareto-optimality.
3. Main idea: use envy-freeness and linearity to show that budgets must be (approximately) equal.

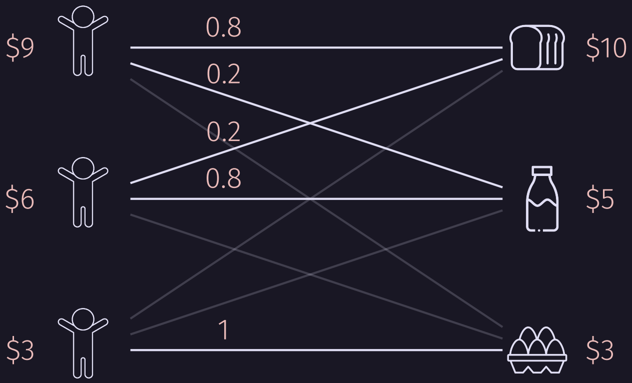
Lemma (Optimal Bundles)

We can find budgets b and prices p, so that for every agent i, x_i is an optimum solution to

$$\begin{aligned} \max \quad & u_i \cdot x_i \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1, \\ & p \cdot x_i \leq b_i, \\ & x_i \geq 0. \end{aligned}$$

≈ Second Welfare Theorem, get prices by setting up correct primal and dual LPs

Step 1 Done



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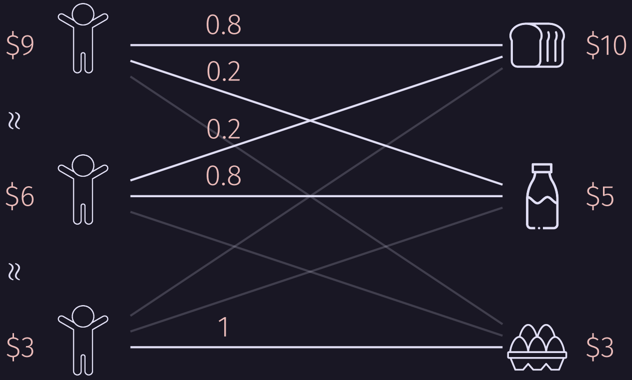

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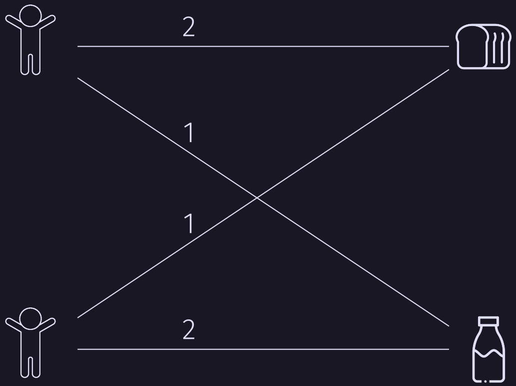
Envy-Freeness and Equal Budgets



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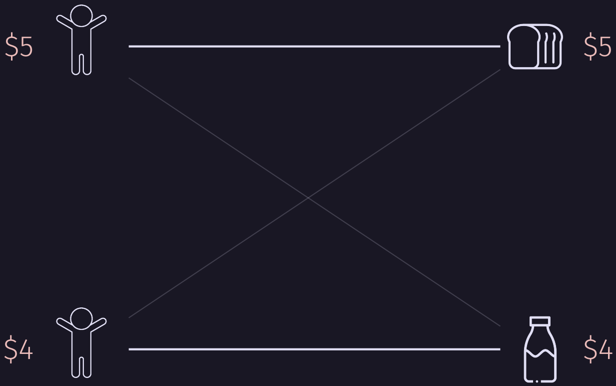
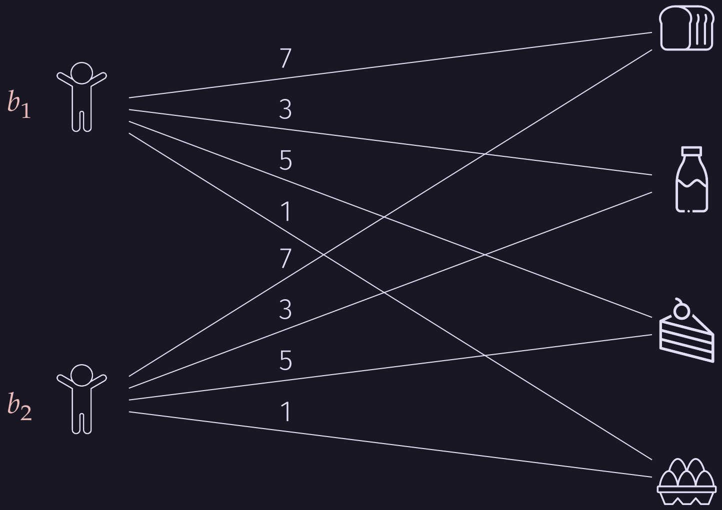


Equal Utilities Imply Equal Budgets



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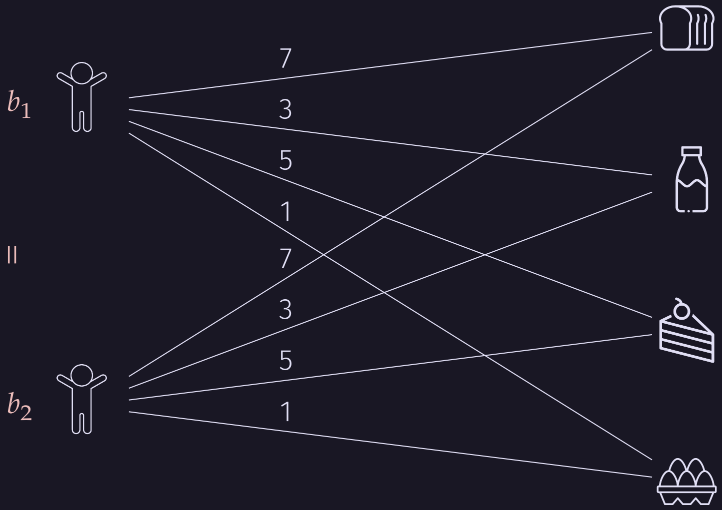

Key Idea 1: Equal Budgets From Envy-Freeness

Lemma

Let $i, i' \in A$ be two agents that agree on all utilities. Then $b_i = b_{i'}$.

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Proof. Suppose $b_i > b_{i'}$. Then i gets a better bundle than i' due to non-satiation. i' agrees that i's bundle is better: envy! □

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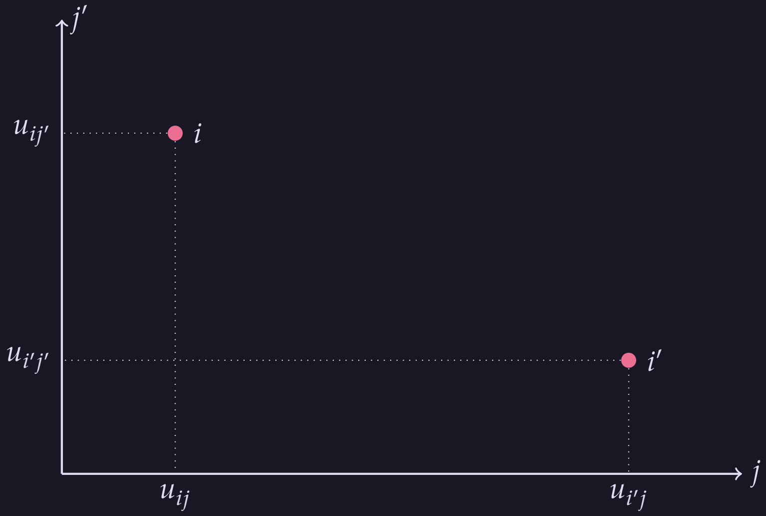
Lemma

Let $i, i' \in A$ be such that utilities agree up to one good where they differ by at most ϵ. Then $|b_i - b_{i'}| \leq 5n^2\epsilon$.

Key Idea 2: Interpolation



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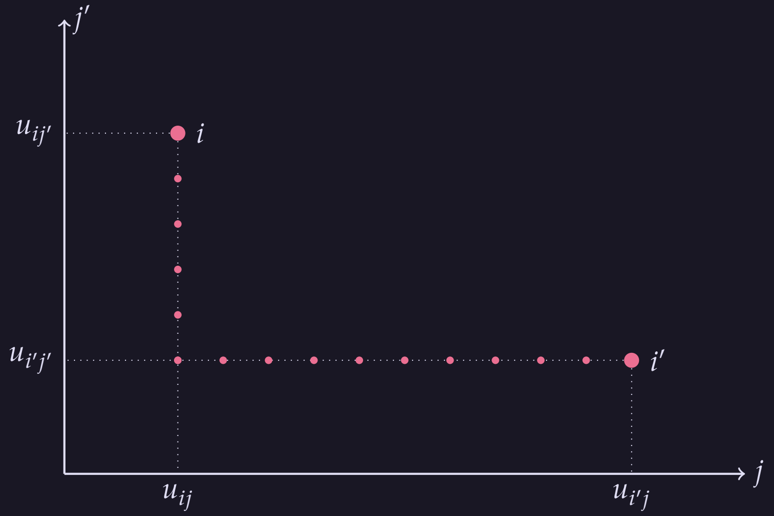
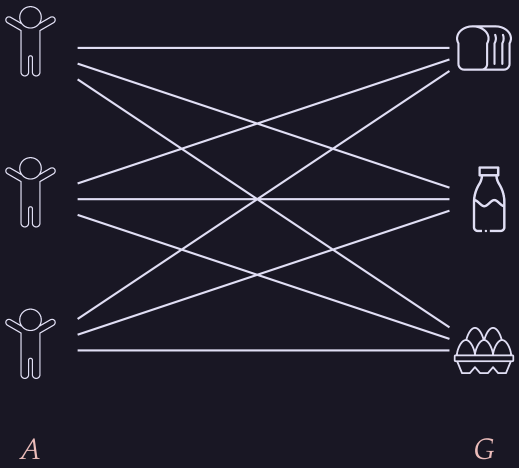


Key Idea 3: Expand the Instance ($k = 4$)



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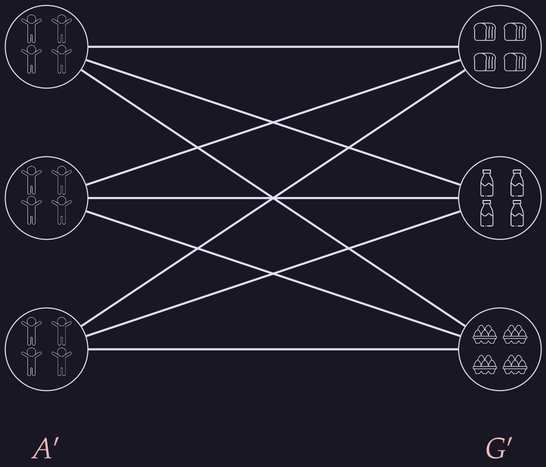
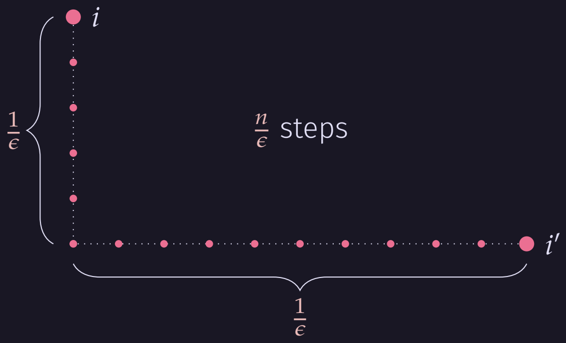


But Does This Help?



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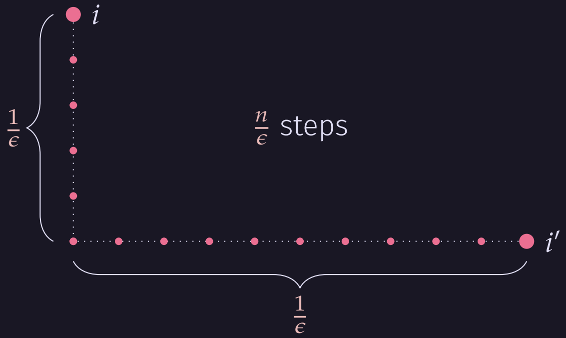

So $|b_i - b_{i'}| \leq \frac{n}{\epsilon} \cdot 5n^2\epsilon = 5n^3$.

But Does This Help?

So $|b_i - b_{i'}| \leq \frac{n}{\epsilon} \cdot 5n^2\epsilon = 5n^3$. Completely useless! ☹

Generalizing to Optimal Bundle Equality

Lemma

Let $i, i' \in A$ such that i and i' agree on which bundles are optimal bundles. Then $b_i = b_{i'}$.

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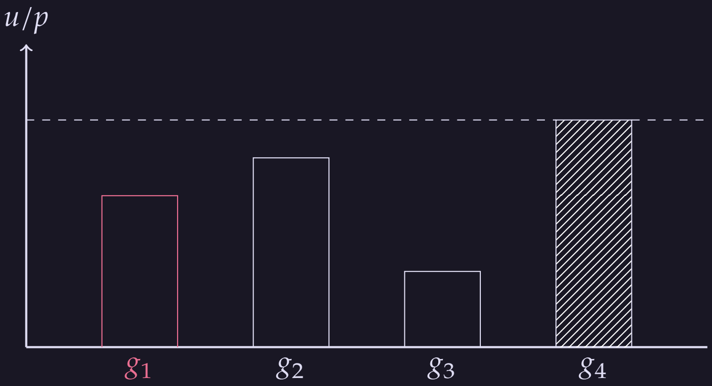
Let $i, i' \in A$ such that i and i' agree on which bundles are optimal bundles. Then $b_i = b_{i'}$.

Caveat:

- In HZ, optimum bundles depend on utilities, prices, and the budget of the agent.
- For the lemma, agents must agree on the optimum bundles at all possible budgets.

Key Idea 4: Optimal Bundles Rarely Change



Without matching constraint: bundles only change when critical bang per buck treshhold is reached.

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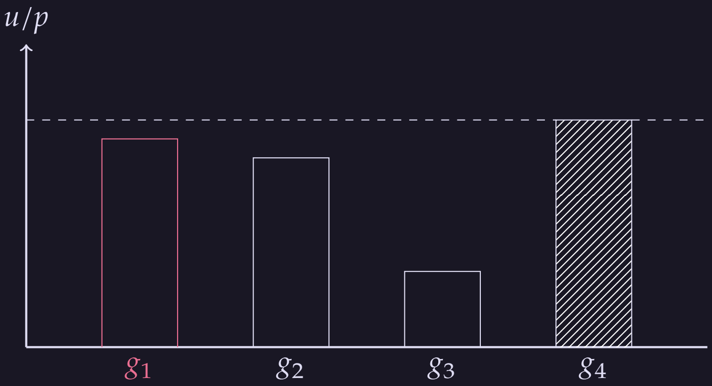

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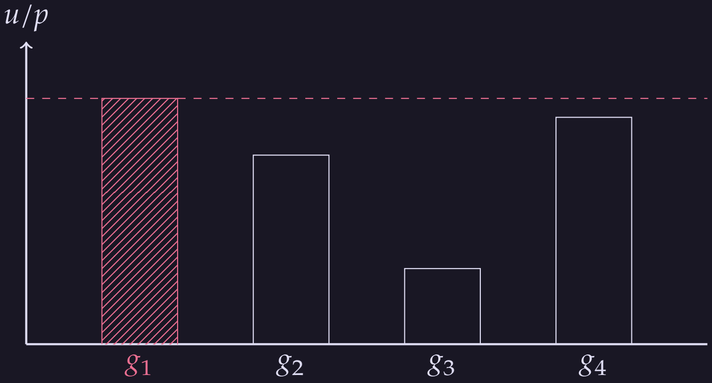

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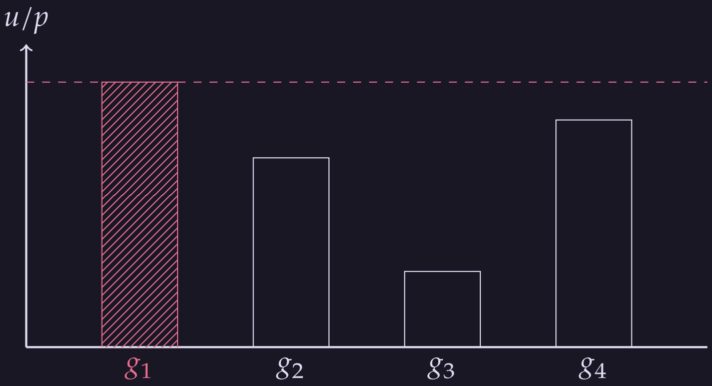

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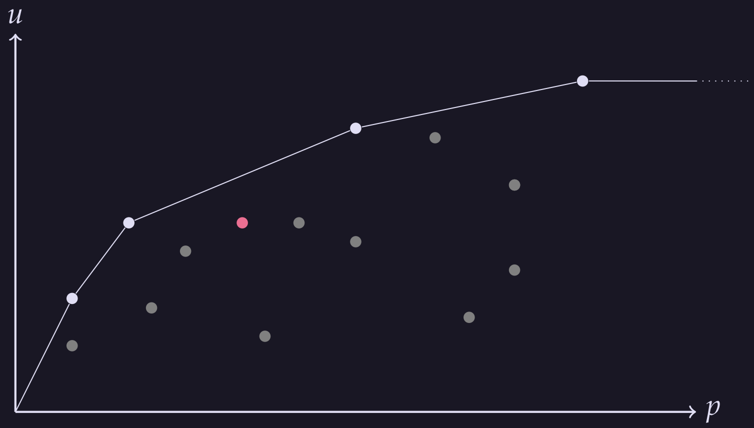
Optimal Bundles in HZ

Recall: x_i is an optimum solution to

$$\begin{aligned} \max \quad & u_i \cdot x_i \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1, \\ & p \cdot x_i \leq t, \\ & x_i \geq 0. \end{aligned}$$

for $t = b_i$.

Key Idea 5: Geometry of Optimal Bundles in HZ



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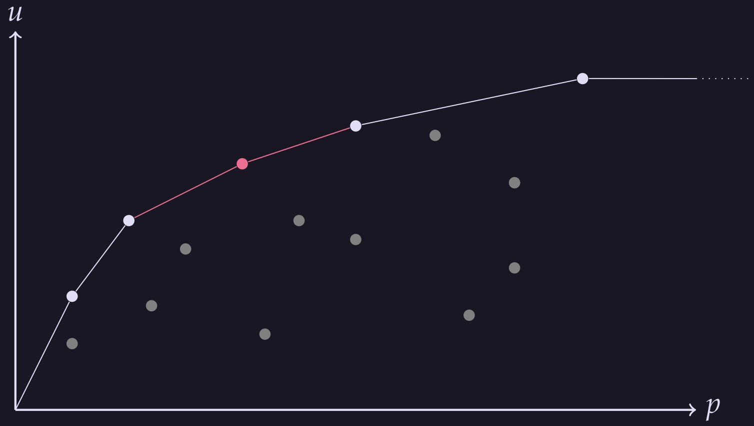
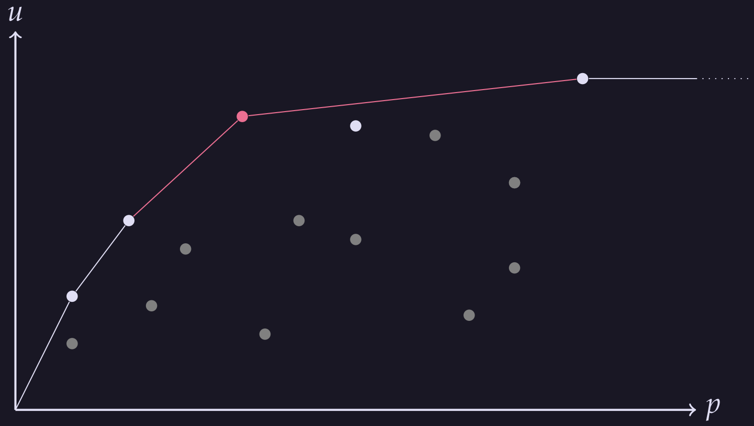


Key Idea 5: Geometry of Optimal Bundles in HZ



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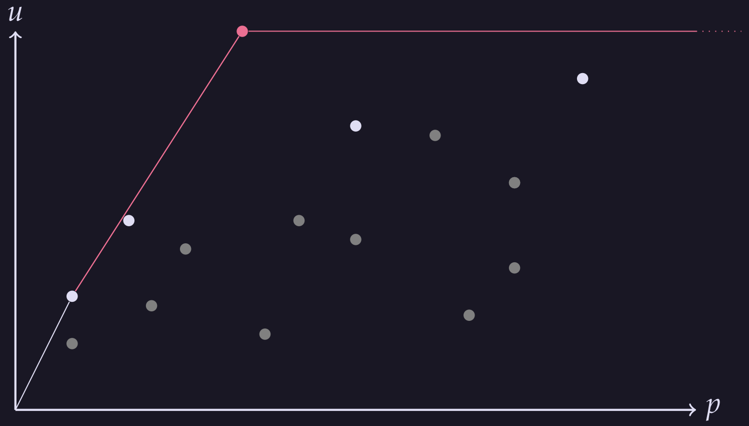

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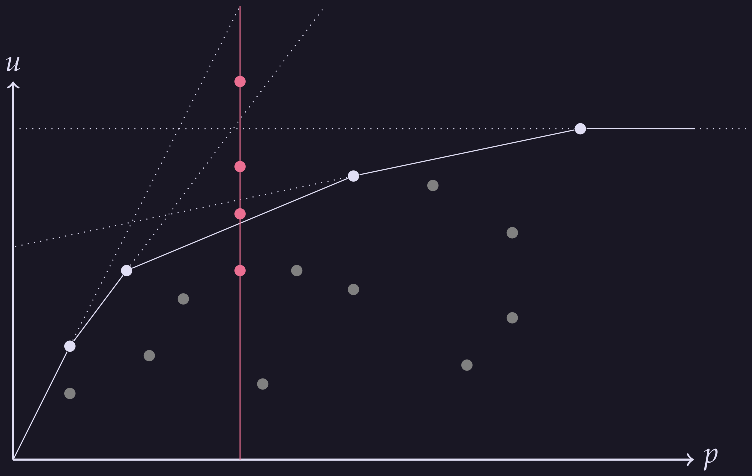

Lemma

Let $i, i' \in A$, then $|b_i - b_{i'}| \leq 5\epsilon n^4$.

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Proof. Between two agents, at most $2n^2$ changes can happen. Each contributes at most $5\epsilon n^2$. □

Theorem

If $\epsilon \leq \frac{1}{5n^5}$ and $k = \frac{n^3}{\epsilon}$, then (x, p) is a $\frac{3}{n}$-approximate HZ equilibrium in the original instance.

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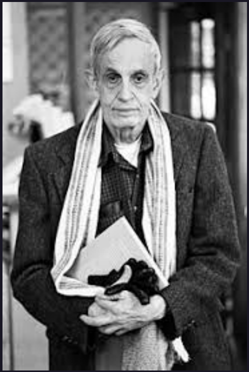
Theorem

The problem of finding an EF+PO allocation in one-sided cardinal-utility matching market is PPAD-hard.

Nash Bargaining

Nash 1950, considered the problem of bargaining:



Nash Bargaining Setup

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- Consider two agents who want to share their vacation homes:
  - Agent 1 has a house in the mountains with utility d_1.
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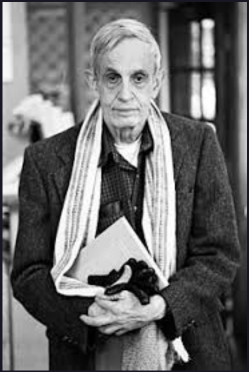

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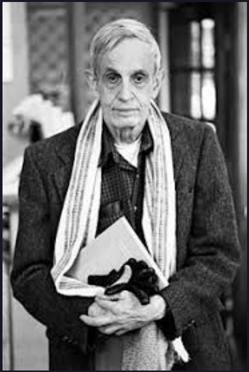
- How should they share?



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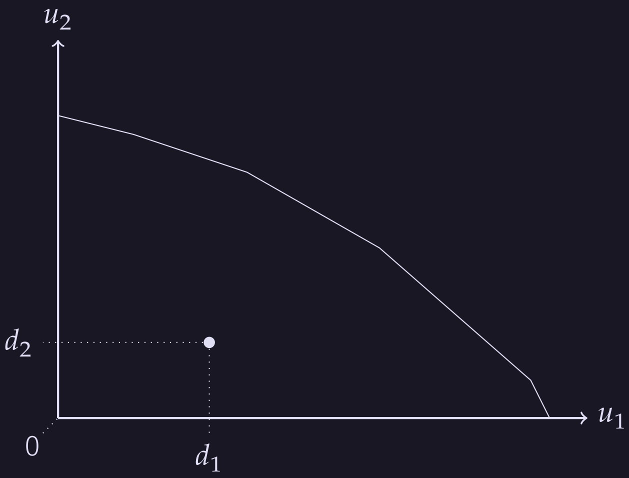

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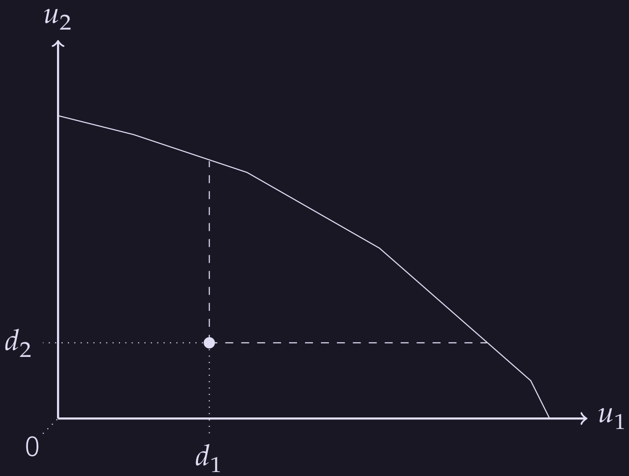

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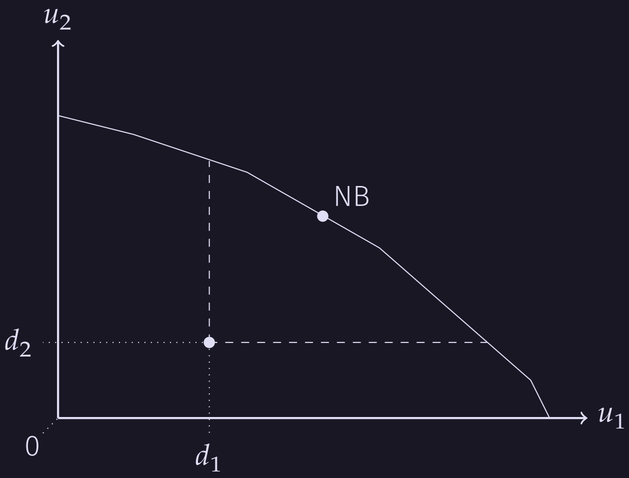

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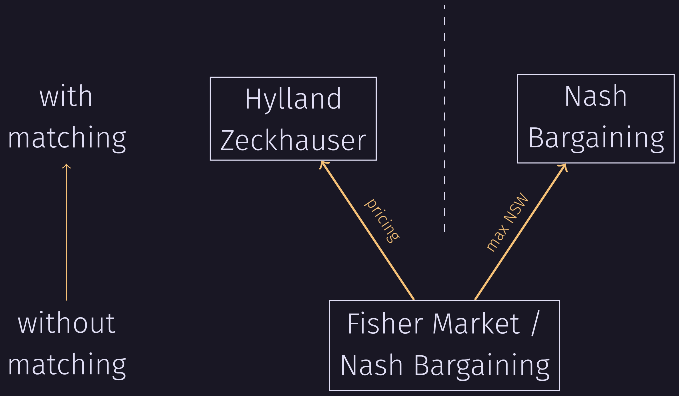
⇒ convex program!

Nash Bargaining CP for Matchings

$$\begin{aligned} \max \quad & \sum_{i \in A} \log(u_i \cdot x_i) \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1 \quad \forall i \in A, \\ & \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\ & x_{ij} \geq 0 \quad \forall i \in A, j \in G. \end{aligned}$$

⇒ Solvable in polynomial time, efficient in practice (Panageas, Tröbst, Vazirani 2024).

Nash Bargaining and EG



Nash Bargaining as an Alternative

Vazirani 2012: NB mechanism for linear Arrow Debreu market

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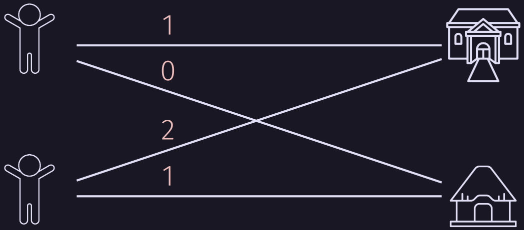
Hosseini, Vazirani 2022: NB mechanism for matching markets

- Tractable alternative to HZ
- Practical algorithms based on convex programming (Frank-Wolfe, cutting planes)
- Computational experiments up to $n = 20000$

Unfortunately, not envy-free:



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Unfairness of Nash Bargaining

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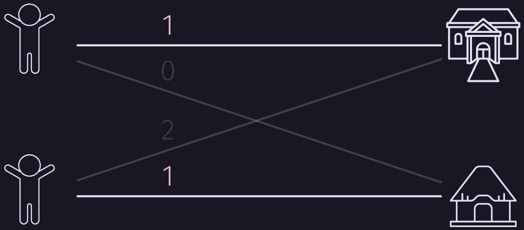

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> 1 for ϵ small enough. □

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Problem

Compute α-EF and 1-PO fractional matchings with $\alpha < 2$.

Thank You!