Cardinal-Utility Matching Markets and Online Matching

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PhD Defense

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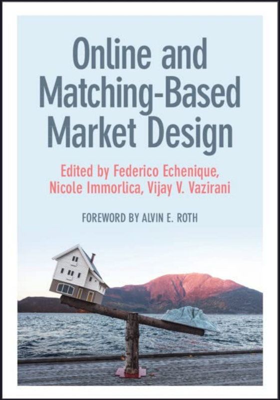

Central themes of my thesis:

- Design mechanisms to find matchings among agents and goods (or other agents)
- Achieve desirable properties (fairness, efficiency, incentive compatibility, etc.)
- Polynomial time algorithms

Matching Markets

Ride-sharing

Delivery

Vacation rental

Ad markets

Focus of my thesis: markets without money, e.g.

- Resident matching
- Kidney donor exchange
- School choice
- National park lotteries

Markets without Money

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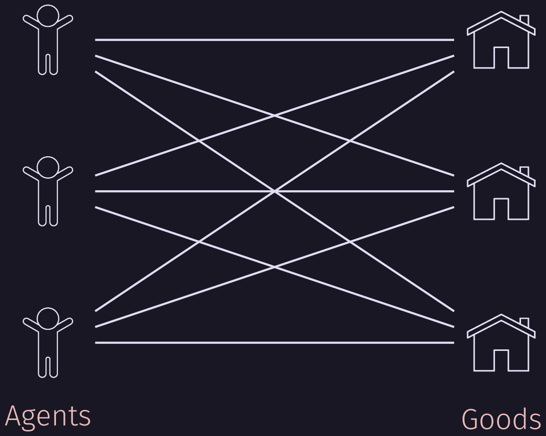
Without money is often necessary but makes things harder!

Overview

- Part I: Cardinal-utility matching markets
  - Markets with endowments
  - Envy-freeness and Pareto-optimality
  - Efficient algorithms for Nash bargaining
- Part II: Online matching
  - Online matching with high probability
  - Online hypergraph matching

Part I: Cardinal-Utility Matching Markets

One-Sided Matching Market



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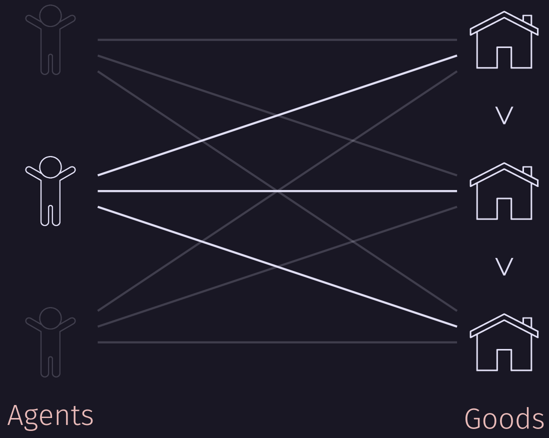

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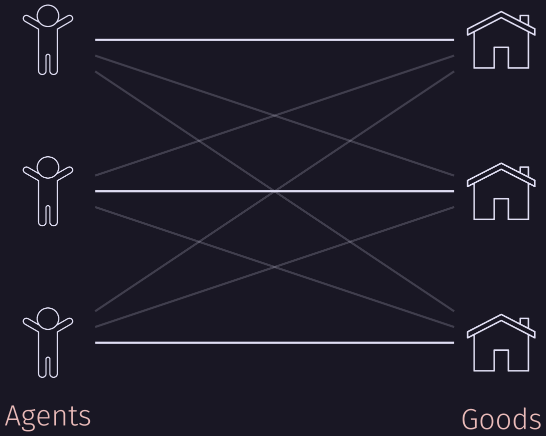

Formal Model

Given

- set A of n agents,
- set G of n goods,
- preferences for each agent over the goods.

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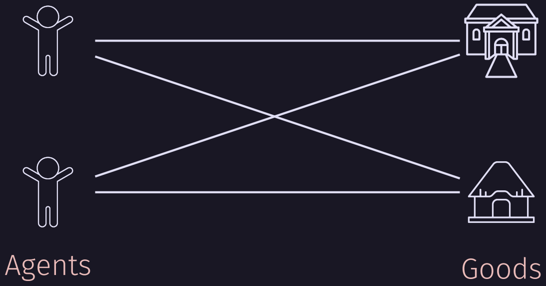
- preferences for each agent over the goods.

Goal:

- Find a perfect matching of agents to goods,
- achieving desirable game-theoretic properties,
- in polynomial time.

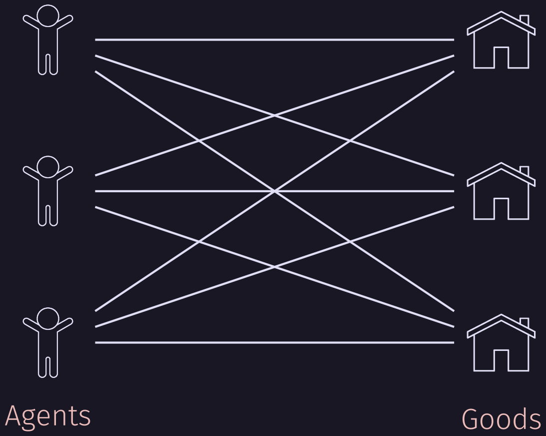
Caveat: Must Allow Lotteries

Cannot achieve fairness without lotteries:



Only fair allocation: run a lottery!

Ordinal vs Cardinal



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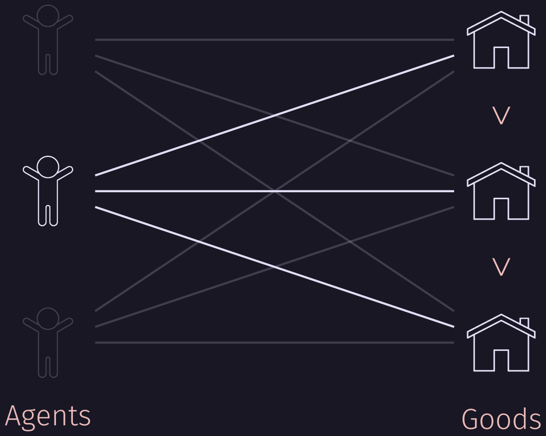

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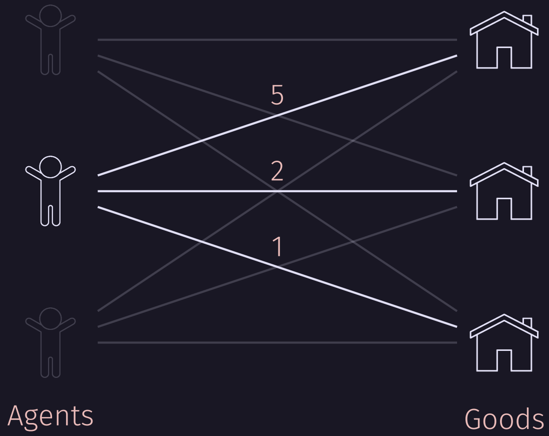

Why Cardinal?

Ordinal preferences have some advantages:

- Easier to elicit
- Simple, efficient algorithms
- Strategyproofness

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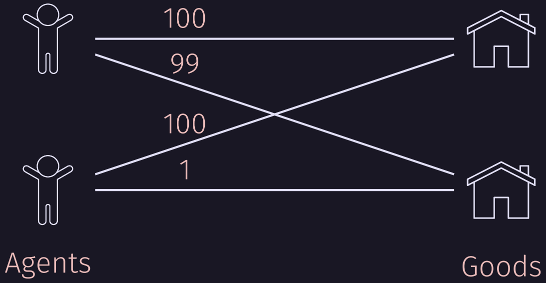
Problem: efficiency!

Hard to be efficient without the cardinal information:



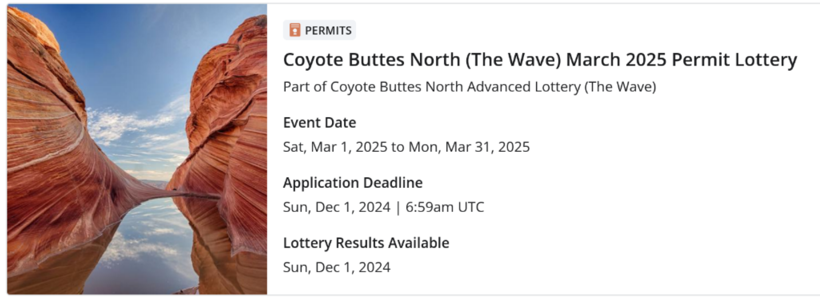
Theorem (Immorlica et al.)

There are instances with n agents and goods such that:

- *all agents agree on the order of the goods,*
- *there is a lottery which improves the utility of every agent by a factor of $\log n$ compared to the uniform lottery.*

⇒ ordinal mechanisms are $\log n$ Pareto inefficient!

Real Example: National Park Lotteries



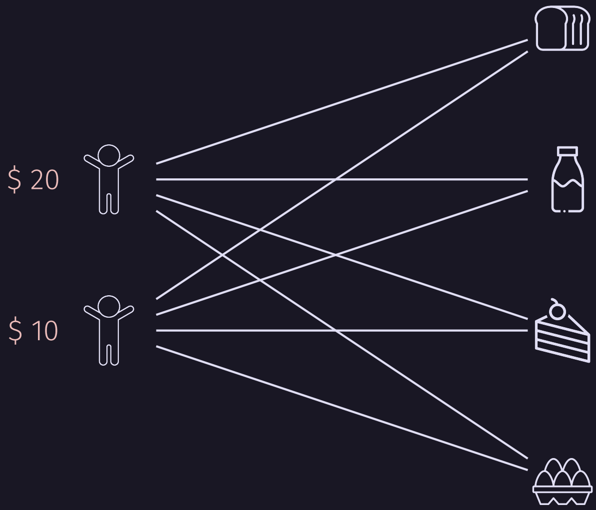
- Goods = days in March
- Each agent can pick three days (modelled via $\{0, 1\}$ utilities)

A story in four acts:

1. The Hylland Zeckhauser Mechanism
2. Challenges and Hardness of HZ
3. Envy-Freeness and Pareto-Optimality
4. Nash Bargaining as an Alternative

1. The Hylland Zeckhauser Mechanism

Fisher Market



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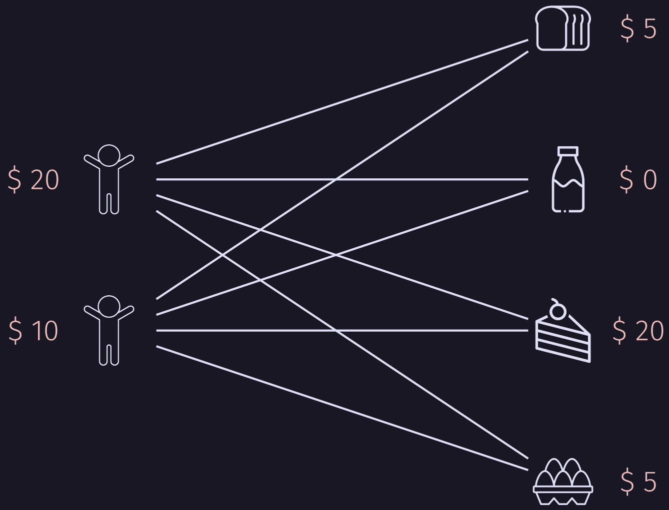

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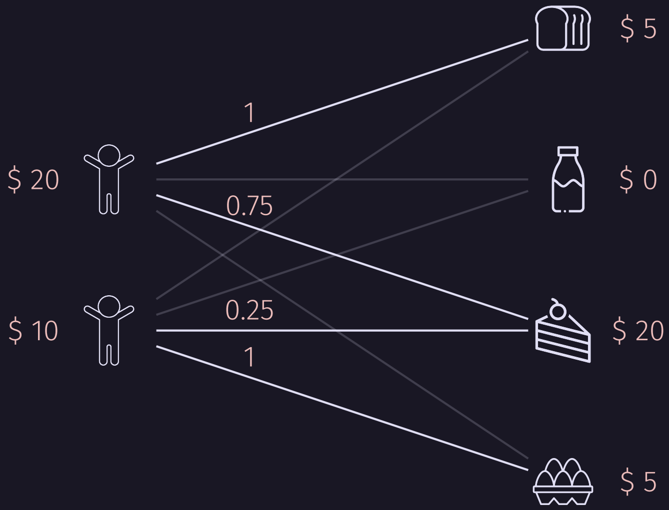

Linear Fisher Market Model

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- utilities $(u_{ij})_{i \in A, j \in G}$.

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Definition (Fisher Market Equilibrium)

A Fisher market equilibrium consists of an allocation $(x_{ij})_{i \in A, j \in G}$ and non-negative prices $(p_j)_{j \in G}$ such that

- every agent spends their budget on a utility-maximizing bundle,
- the market clears.

Hylland Zeckhauser Mechanism

Hylland and Zeckhauser (1979) give pricing-based mechanism for cardinal-utility matching market:

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⇒ Intuitively: HZ ≈ Fisher market + matching + rounding

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1. x is a fractional perfect matching.
2. No agent overspends, i.e. $p \cdot x_i \leq 1$.
3. Every agent gets optimum bundle, i.e. $u_i \cdot x_i = \max\{u_i \cdot y \mid \sum_{j \in G} y_j = 1, p \cdot y \leq 1\}$.

Properties of HZ Equilibria

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Theorem (Hylland, Zeckhauser 1979)

HZ equilibria always exist (proof via non-constructive Kakutani's fixed point theorem).

How do you find an HZ equilibrium?

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Theorem (Devanur, Papadimitrious, Saberi, Vazirani 2002)

Can find Fisher market equilibria in polynomial time using combinatorial, flow-based algorithm. Always finds rational equilibrium.

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Conjecture

HZ algorithm = DPSV + matching? Should be doable!

2. Challenges and Hardness of HZ

Theorem (Alaei, Khalilabadi, Tardos 2017)

There is an algorithm based on algebraic cell decomposition which checks $> n^{5n^2}$ cells.

Theorem (Vazirani, Yannakakis 2020)

There is a polynomial time algorithm for $\{0,1\}$ utilities.

⇒ galactic running time or restrictive utilities...

Theorem (Vazirani, Yannakakis 2020)

There are instances of HZ in which there is a unique equilibrium with irrational allocations and prices!

$\Rightarrow$ rules out exact, combinatorial algorithm

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Theorem (Vazirani, Yannakakis 2020)

HZ is in FIXP, approximate HZ is in PPAD.

Hardness

Theorem (Chen, Chen, Peng, Yannakakis 2022)

The problem of computing an ϵ-approximate HZ-equilibrium is PPAD-hard for $\epsilon = 1/n^c$ for any constant $c > 0$.

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The problem of computing an ϵ-approximate HZ-equilibrium is PPAD-hard for $\epsilon = 1/n^c$ for any constant $c > 0$.

$\Rightarrow$ computing HZ-equilibria is as hard as

- computing general Nash-equilibria,
- computational versions of Kakutani's / Brouwer's fixed-point theorems.

☹

Challenge

HZ is highly specific (one-sided, linear) but general equilibrium theory has much broader applications.

⇒ some results in chapter “Markets with Endowments”, won’t cover these today

4. Pareto-Optimality and Envy-Freeness

Question

Recall that HZ is

- *Pareto-optimal (PO): can't improve one agent without hurting another.*
- *Envy-free (EF): no agent prefers another agents' lottery odds to their own.*

Central Question

Question

Recall that HZ is

- *Pareto-optimal (PO): can't improve one agent without hurting another.*
- *Envy-free (EF): no agent prefers another agents' lottery odds to their own.*

Can we find an envy-free (EF) and Pareto-optimal (PO) allocation in polynomial time?

EF+PO and HZ are quite different:

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1. HZ may have only irrational solutions, but there are always rational EF+PO solutions

Main Result

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2. HZ little structure (fixed point), but EF+PO is polyhedral

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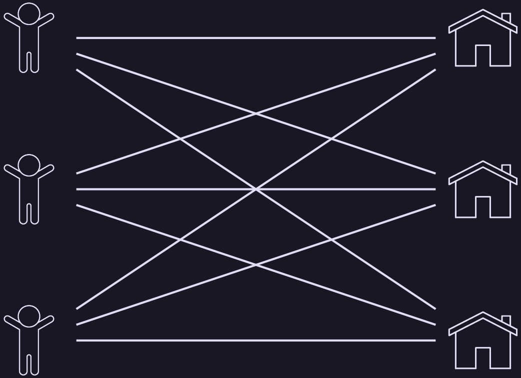
Theorem (Tröbst, Vazirani 2024)

Finding an EF+PO allocation is PPAD-hard.

Strategy: polynomial reduction of approximate HZ to EF+PO



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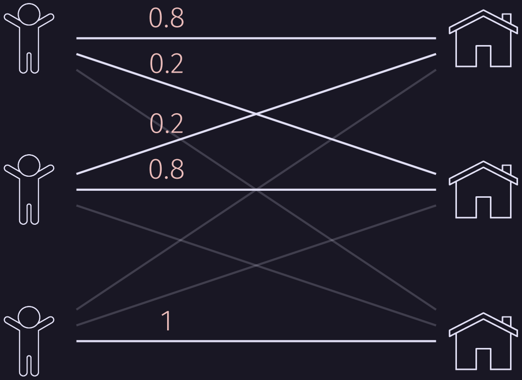

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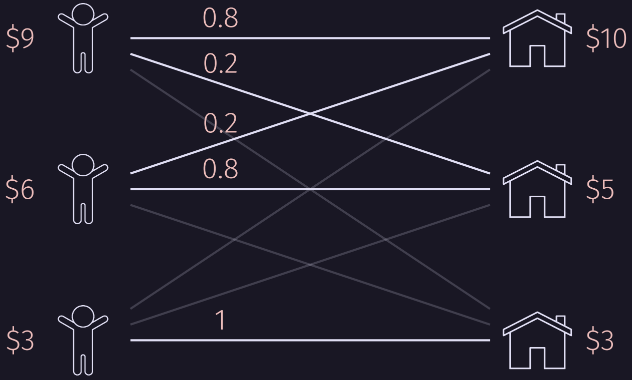

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3. Main idea: use envy-freeness and linearity to show that budgets must be (approximately) equal.

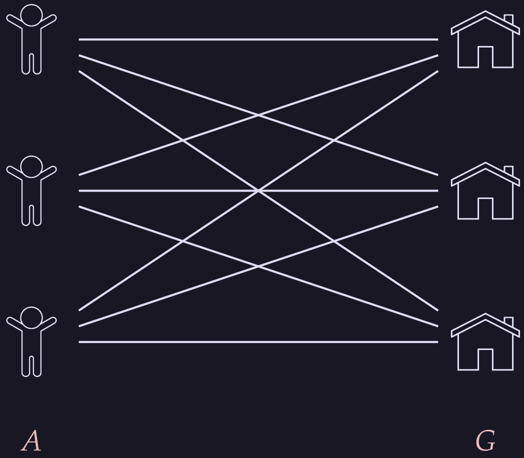
Lemma (Optimal Bundles)

We can find budgets b and prices p, so that for every agent i, x_i is an optimum solution to

$$\begin{aligned} \max \quad & u_i \cdot x_i \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1, \\ & p \cdot x_i \leq b_i, \\ & x_i \geq 0. \end{aligned}$$

≈ Second Welfare Theorem, get prices by setting up correct primal and dual LPs

Idea 1: Expand the Instance ($k = 4$)



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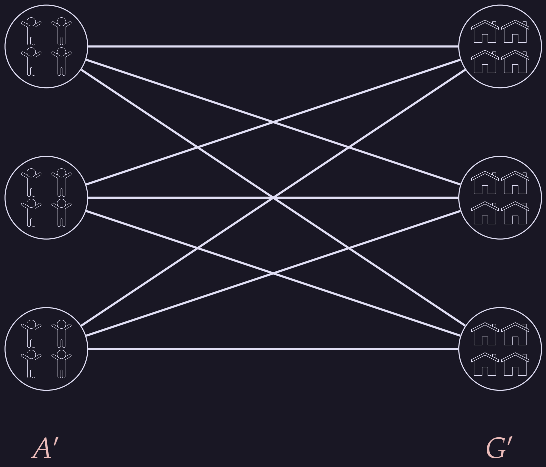

Idea 2: Equal Budgets From Envy-Freeness

Lemma

Let $i, i' \in A$ be two agents that agree on all utilities. Then $b_i = b_{i'}$.

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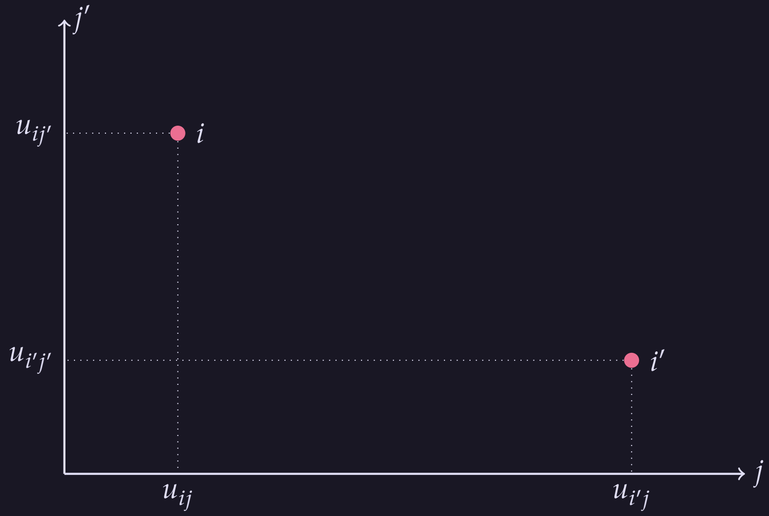
Lemma

Let $i, i' \in A$ be such that utilities agree up to one good where they differ by at most ϵ. Then $|b_i - b_{i'}| \leq 5n^2\epsilon$.

Idea 3: Interpolation



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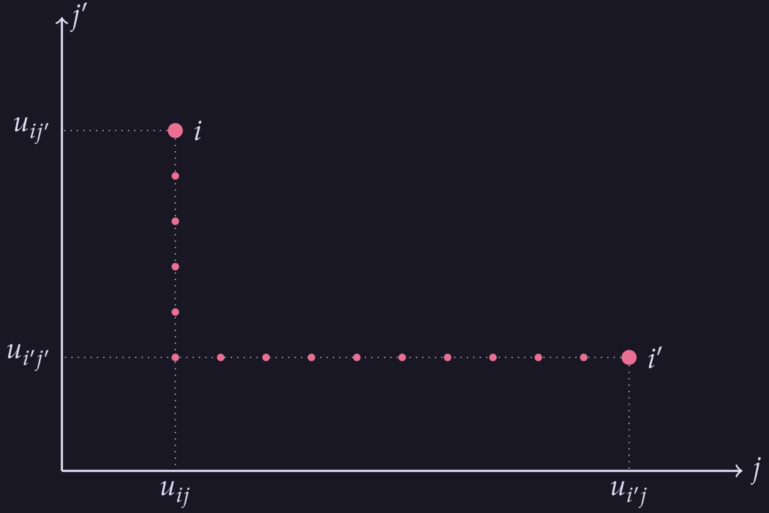

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Answer: Up to $\frac{n}{\epsilon}$.

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Completely useless! ☹

Generalizing to Optimal Bundle Equality

Lemma

Let $i, i' \in A$ such that i and i' agree on which bundles are optimal bundles. Then $b_i = b_{i'}$.

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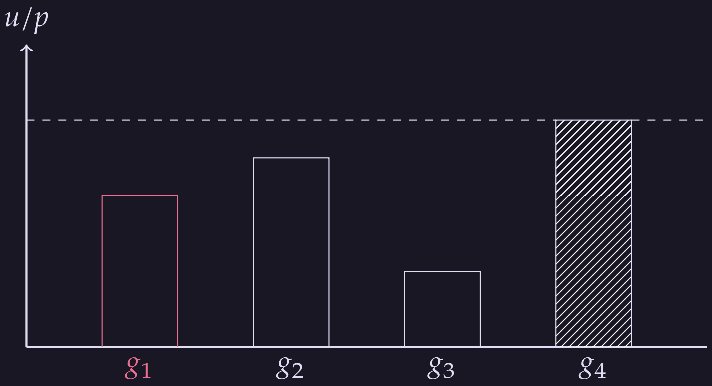
Caveat:

- In HZ, optimum bundles depend on utilities, prices, and the budget of the agent.
- For the lemma, agents must agree on the optimum bundles at all possible budgets.

Key Idea: Optimal Bundles Rarely Change



Without matching constraint: bundles only change when critical bang per buck treshhold is reached.

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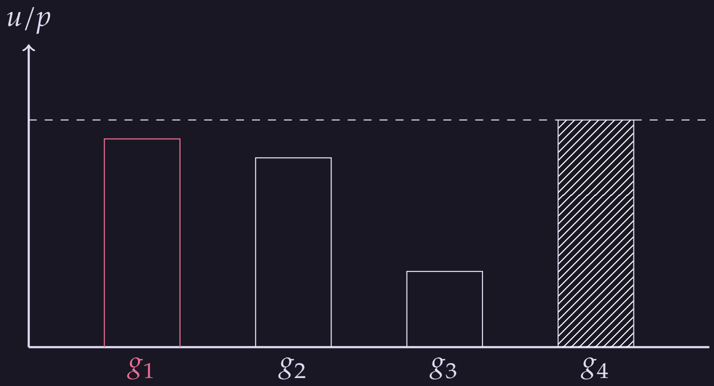

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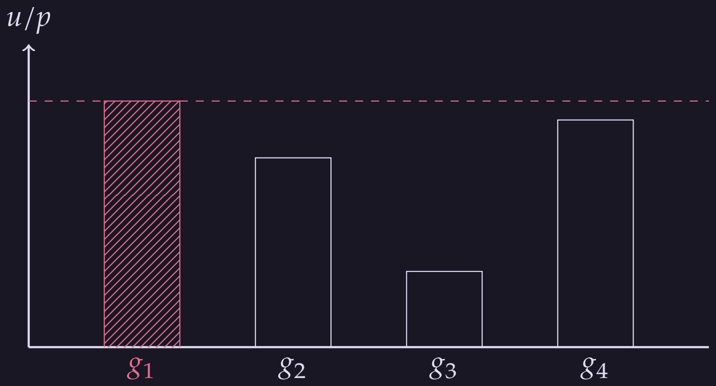

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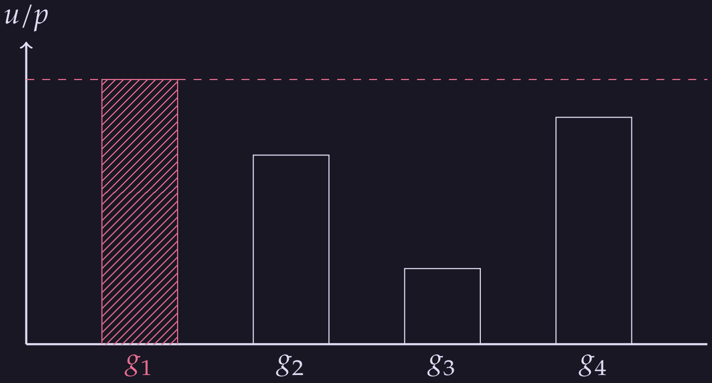

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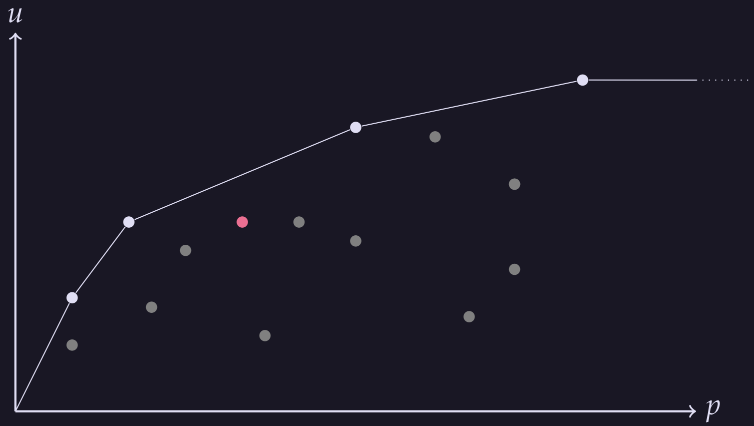
Optimal Bundles in HZ



In HZ: more complex characterization of optimal bundles.

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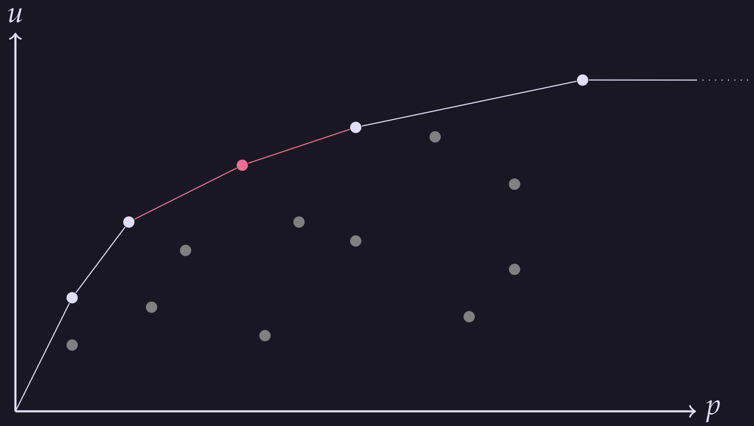

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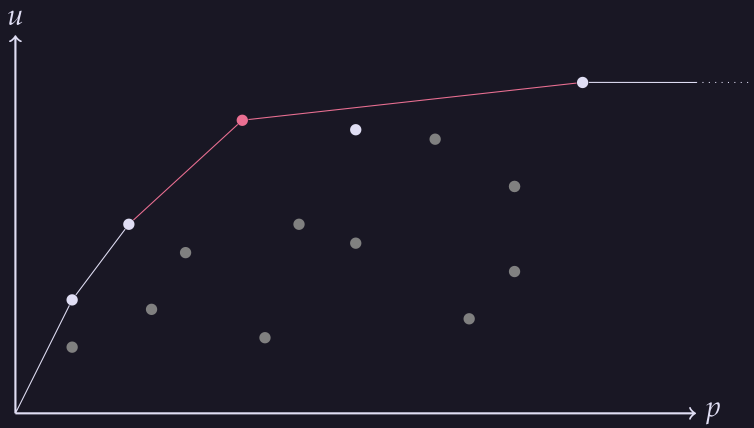

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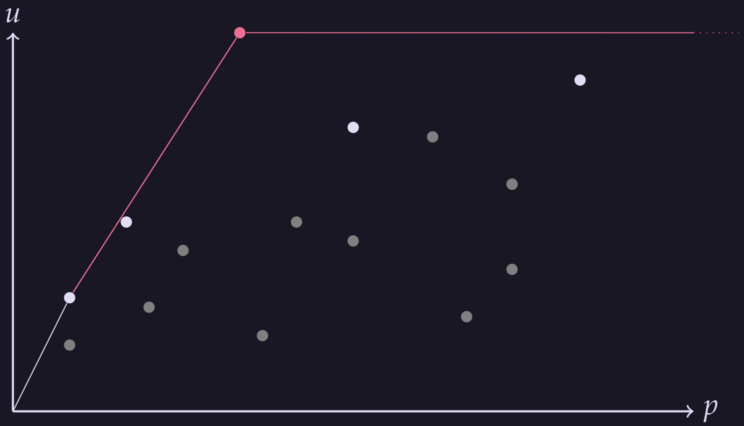

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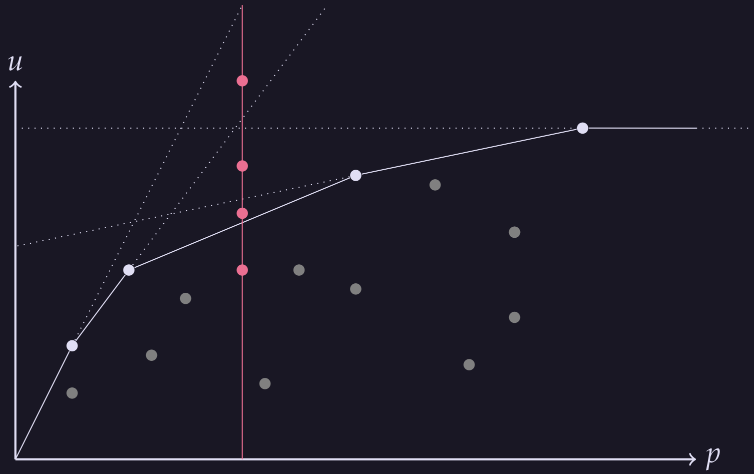

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Proof. Between two agents, at most $2n^2$ changes can happen. Each contributes at most $5\epsilon n^2$. □

Bringing It Together II

Theorem

If $\epsilon \leq \frac{1}{5n^5}$ and $k = \frac{n^3}{\epsilon}$, then (x, p) is a $\frac{3}{n}$-approximate HZ equilibrium in the original instance.

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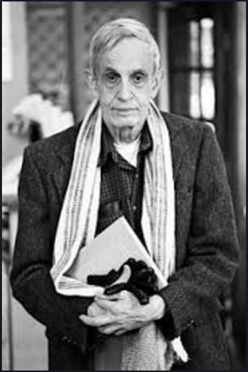
Theorem

The problem of finding an EF+PO allocation in one-sided cardinal-utility matching market is PPAD-hard.

4. Nash Bargaining

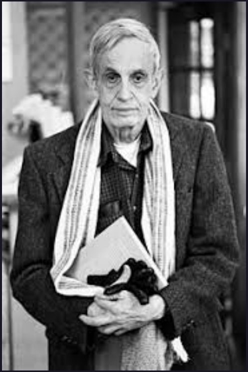
Nash 1950, considered the problem of bargaining:



Nash Bargaining Setup

Nash 1950, considered the problem of bargaining:

- Consider two agents who want to share their vacation homes:
  - Agent 1 has a house in the mountains with utility d_1.
  - Agent 2 has a house on the beach with utility d_2.



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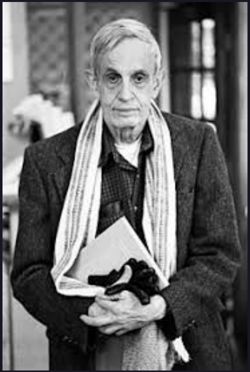
- How should they share?



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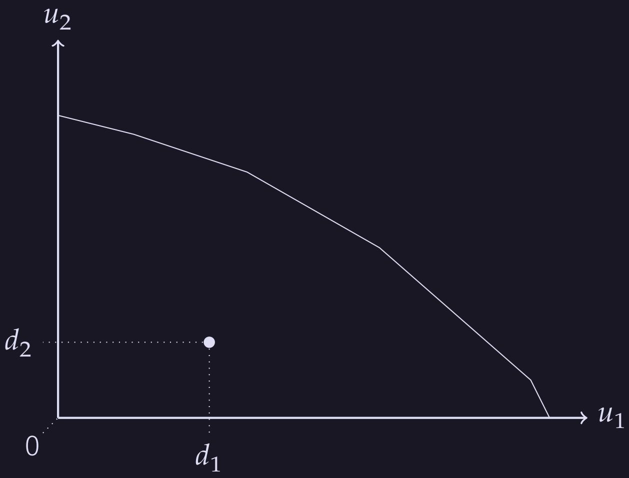

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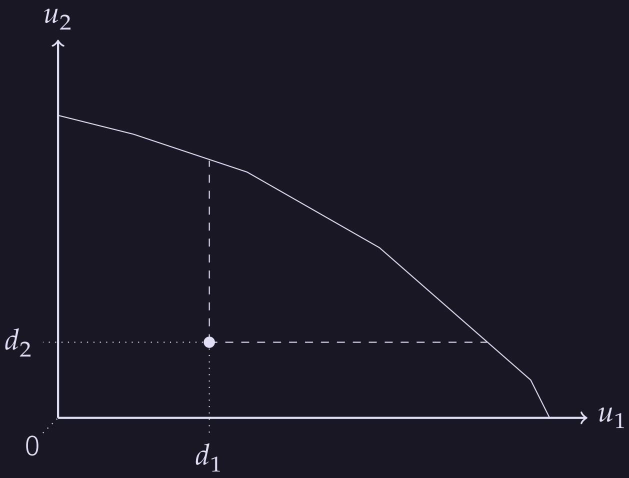

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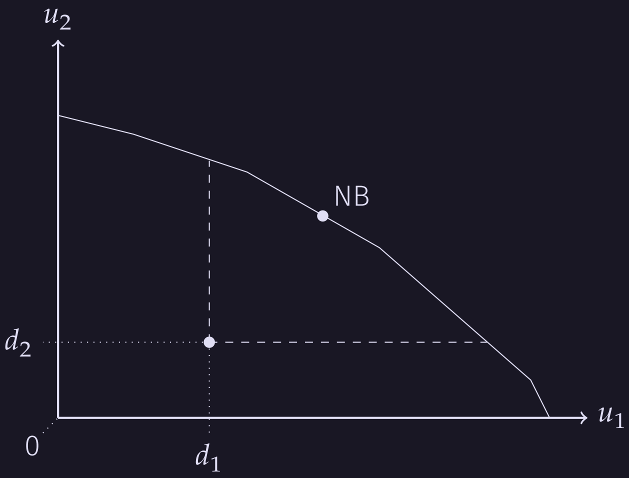

Theorem (Nash 1950)

Let U, set of utility vectors, be convex. Then

- *there is a unique point satisfying Pareto-optimality, symmetry, invariance under affine transformations, and independence of irrelevant alternatives.*
- *it is the maximizer of $\prod_{i \in A}(u_i - d_i)$ for $u \in U$.*

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Nash-Bargaining and Pricing

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Vazirani 2012: Nash-bargaining-based mechanism for linear Arrow Debreu market

- Nash bargaining is rational convex program
- Nash bargaining and pricing not the same
- Combinatorial, strongly polynomial time algorithm

Results (Hosseini, Vazirani 2022)

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Results (Hosseini, Vazirani 2022)

Extends to many other models inspired by general equilibrium theory.

Theorem (Panageas, Tröbst, Vazirani 2022)

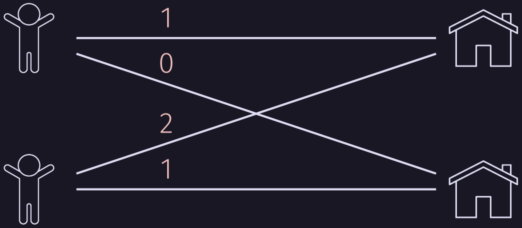
We can compute an ϵ-approximate Nash bargaining solution after $O\left(\frac{n \log n}{\epsilon^2}\right)$ iterations of a multiplicative-weights algorithm. Each iteration can be carried out in $O(n^2)$ time.

Theorem (Panageas, Tröbst, Vazirani 2022)

We can compute an ϵ-approximate Nash bargaining solution after $O\left(\frac{n^3 \kappa^2}{\epsilon}\right)$ iterations of a conditional gradient algorithm. Each iteration consists of computing a max-weight bipartite matching ($O(n^3)$ time).

Unfairness of Nash Bargaining

Unfortunately, not envy-free and not strategy-proof:



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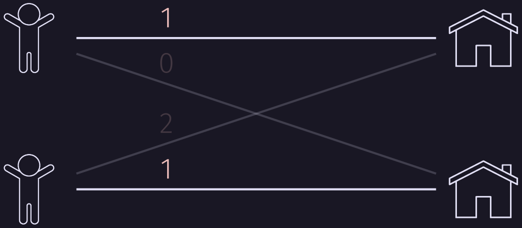

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Theorem (Tröbst, Vazirani 2024)

The Nash-bargaining-based mechanism is 2-approximately envy-free.

Theorem (Tröbst, Vazirani 2024)

The Nash-bargaining-based mechanism is 2-approximately incentive-compatible.

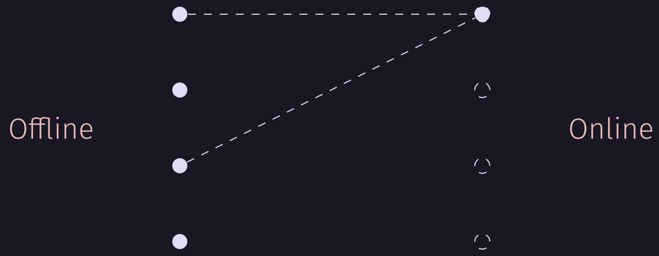
Conclusion

Nash bargaining is a practical HZ alternative for one-sided cardinal-utility matching markets.

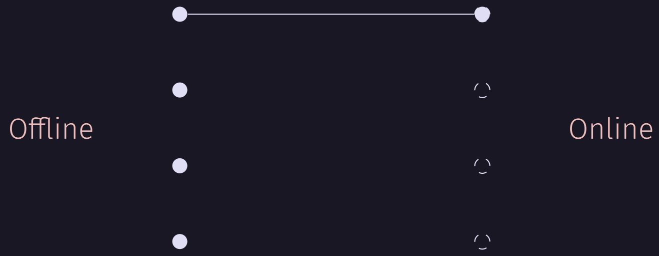
Part II: Online Matching


Offline
Online

Online Bipartite Matching



Online Bipartite Matching



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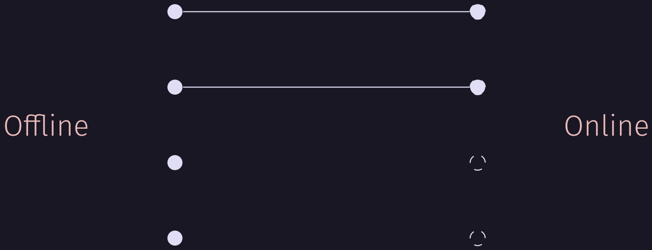

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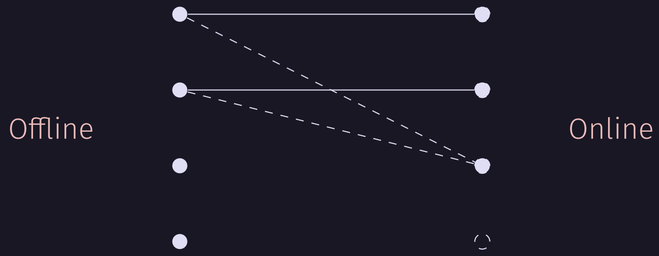

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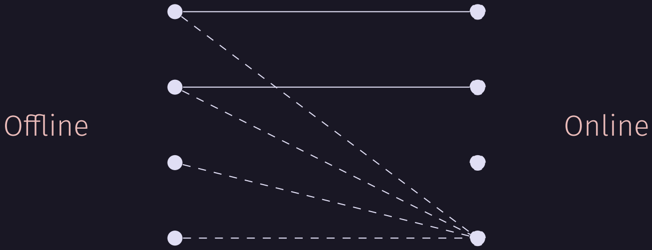

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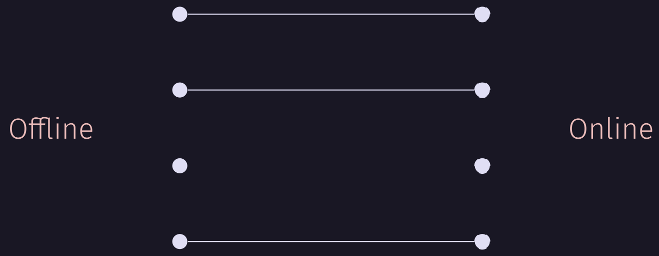

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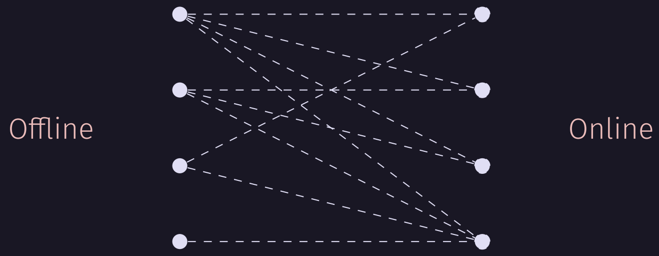

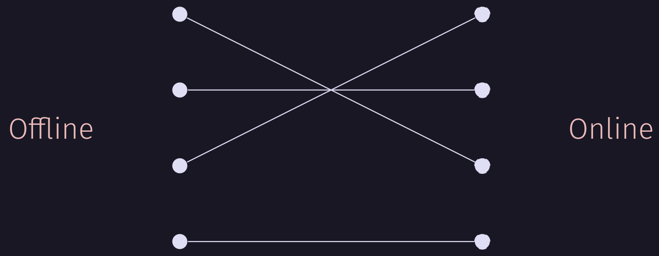
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Online Bipartite Matching II

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- Online vertices arrive one by one in adverserial order.
- The algorithm must irrevocably and immediately match revealed online vertices.
- The goal is to maximize the competitive ratio, i.e.

$$\frac{|M_{\text{online}}|}{\text{OPT}_{\text{offline}}}.$$

Typical applications include online advertising or ride hailing.

(highly simplified model)

Theorem (Karp, Vazirani, Vazirani 1990)

The GREEDY algorithm (match whenever possible) is $1/2$-competitive. $1/2$-competitive is best possible for deterministic algorithms.

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The randomized RANKING algorithm is $(1 - 1/e)$-competitive in expectation. $(1 - 1/e)$-competitive in expectation is best possible for randomized algorithms.

Online Matching with High Probability

The Power of Randomized Algorithms

Many problems have more natural, efficient, or better algorithms using randomization:

- Quicksort
- Miller-Rabin primality test
- Hashing
- Polynomial identity testing
- Perfect matching on parallel machines
- Many online algorithms!

Expectation vs. Concentration

Usually we analyze in expectation or showing non-zero probability of success.

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- Fewer know: $\mathbb{P}[C > c_0 \cdot n \log n] < \frac{1}{n}$ for some c_0.
- But did you know:

$$\mathbb{P}[|C/\mathbb{E}[C] - 1| > \epsilon] < n^{-2\epsilon(\ln \ln n - \ln(1/\epsilon) + O(\ln \ln \ln n))}$$

Usefulness of Concentration Results

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Problem

Online algorithms cannot be boosted!

Question

Does the competitive ratio of Ranking hold with high probability or just in expectation?

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Theorem (Mihail, Tröbst 2021)

Let M be the matching generated by Ranking, then

$$\mathbb{P}\left[|M| < \left(1 - \frac{1}{e} - \alpha\right)\mathrm{OPT}\right] < e^{-2\alpha^2 \mathrm{OPT}}.$$

Theorem (Mihail, Tröbst 2021)

For the Fully Online Matching Problem, we have

$$\mathbb{E}[|M| < (\rho - \alpha)\mathrm{OPT}] < e^{-\alpha^2 \mathrm{OPT}}$$

where M is produced by FULLY ONLINE RANKING *and $\rho \approx 0.521$.*

Theorem (Mihail, Tröbst 2021)

For the Vertex-Weighted Online Bipartite Matching Problem and each $\alpha > 0$, there exists an algorithm such that

$$\mathbb{P}\left[w(M) < \left(1 - \frac{1}{e} - \alpha\right)\mathrm{OPT}\right] < e^{-\frac{1}{50}\alpha^4 \frac{\mathrm{OPT}^2}{\|w\|_2^2}}.$$

Thank You!

Acknowledgments

- The chapter on HZ with endowments is based on "One-Sided Matching Markets with Endowments: Equilibria and Algorithms" which was published in JAAMAS and co-authored with Jugal Garg and Vijay Vazirani.
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