Finding the Right Curve: Optimal Design of Constant Function Market Makers[1]

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[1] M. Goyal, G. Ramseyer, A. Goel, D. Mazieres EC 2023

Pont au Change, Paris

Exchanges

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- Virtual goods (video game items, NFTs, etc.)

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- Standing orders form the order book
- When lowest ask and highest bid cross, a trade is made

Order Book Example

An order book:

202560 for sale starting at $1.19

Buy...

Price	Quantity
$1.19	2
$1.20	8
$1.21	6
$1.22	585
$1.23	1167
$1.24 or more	200792

3557901 requests to buy at $1.17 or lower

Sell...

Price	Quantity
$1.17	4602
$1.16	4728
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(Steam community market)

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- Buyers / sellers may need to wait
- Prices can shift quickly if there are too few standing orders
- Highly centralized

Illiquid Order Book

- With few participants, there are large gaps in the order book
- Such a market is illiquid an inefficient

Price	Quantity
$510.00	3
$427.88	1
$397.00	1
$380.00	1
$379.00	1
$320.51 or less	183

(lowest ask is $700)

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  - Market maker profits from bid-ask spread and/or commission

Definition

A Constant Function Market Maker with trade function $f : \mathbb{R}^2_{\geq 0} \to \mathbb{R}$ holds two assets X and Y. It accepts a trade changing its holdings from (x, y) to (x', y') iff $f(x, y) = f(x', y')$.

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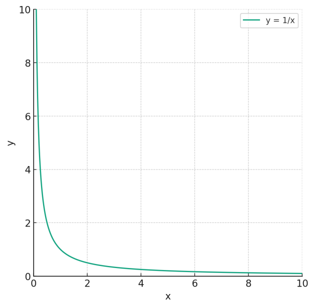
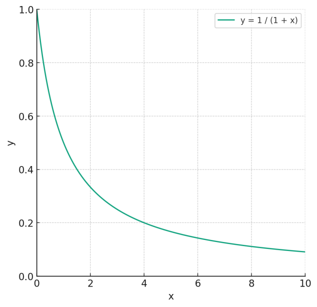
- I will accept \$1.1 for €1 because $99 \cdot 111.1 \approx 100 \cdot 110$
- I will accept \$12 for €10 because $90 \cdot 122 \approx 100 \cdot 110$

Level Sets and Exchange Rates



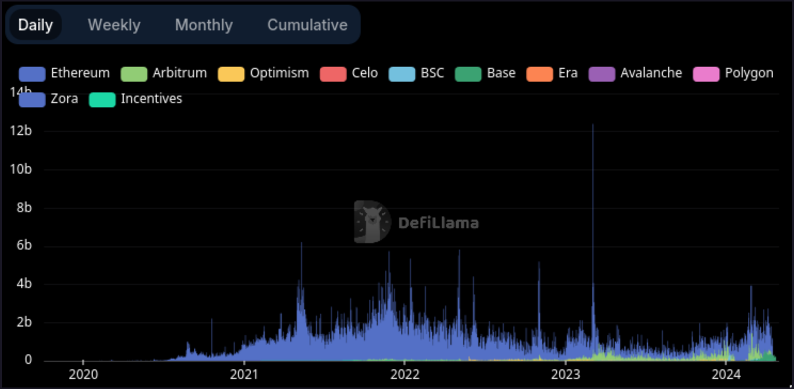
Acceptable holdings



Exchange rate if holding $(1, 1)$

CFMM Success

CFMMs are popular in prediction markets and decentralized finance. Uniswap alone trades over $1 billion per day.



Source: DefiLlama

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$\Rightarrow$ arbitrage

Example:

- $x = €100$ and $y = \$110$
- Exchange rate changes to $€1 = \$1.2$.
- Rational agents will trade with us until we have $y = 1.2x$ because they make risk-free profit (arbitrage)

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- But our original holdings would have been worth \$230!

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Order succeeds if the entire trade can be done within maximum slippage!

Question

What is the best choice of f to maximize chance of trading?

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What is the best choice of f to maximize expected fee revenue minus divergence loss?

Spot Exchange Rate

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Lemma

Under reasonable assumptions on f, p determines $x = \mathcal{X}(p)$ and $y = \mathcal{Y}(p)$.

Definition

The liquidity at exchange rate p is

$$L(p) := \frac{\mathrm{d}\mathcal{Y}(p)}{\mathrm{d}\ln(p)} = p\frac{\mathrm{d}\mathcal{Y}(p)}{\mathrm{d}p}$$

or alternatively

$$L(p) := p\frac{\mathrm{d}\mathcal{X}(p)}{\mathrm{d}\ln(1/p)} = -p^2\frac{\mathrm{d}\mathcal{X}(p)}{\mathrm{d}p}$$

Lemma

We can write:

$$\mathcal{Y}(p) = \int_0^p \frac{L(t)}{t}\,\mathrm{d}t$$

$$\mathcal{X}(p) = \int_p^\infty \frac{L(t)}{t^2}\,\mathrm{d}t$$

Goal

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We need to answer:

Question

What is the probability that a trade fails for some slippage ϵ?

Our model is:

- Fixed reference exchange rate $\hat{p}$
- Agents accept slippage of small ϵ relative to $\hat{p}$
- At each time step, BUY or SELL with equal probability
- All trades have equal size k in Y (not needed)

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$$\mathcal{Y}\left(\frac{\hat{p}}{1+\epsilon}\right) - \mathcal{Y}((1+\epsilon)\hat{p}) = \int_{\hat{p}/(1+\epsilon)}^{\hat{p}(1+\epsilon)} \frac{L(t)}{t}\,\mathrm{d}t$$
$$\approx 2\ln(1+\epsilon)L(\hat{p})$$

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Probability of being on the boundary: $2\,\frac{k}{2\ln(1+\epsilon)L(\hat{p})}$.

Probability of failure on the boundary: $\frac{1}{2}$. $\square$

Theorem

Let $\psi(p_X, p_Y)$ be a distribution over prices (in \$). If agents trade \$1 worth of goods, then expected inefficiency is (proportional to):

$$\int\int_{p_X, p_Y} \frac{\psi(p_X, p_Y)}{p_Y L(p_X/p_Y)} \, \mathrm{d}p_X \, \mathrm{d}p_Y$$

Optimizing over Beliefs

Theorem

The optimal CFMM is given by

$$\begin{aligned}
\min \quad & \int\int_{p_X, p_Y} \frac{\psi(p_X, p_Y)}{p_Y L(p_X/p_Y)} \, \mathrm{d}p_X \, \mathrm{d}p_Y \\
\text{s.t.} \quad & \int_0^{p_0} \frac{L(p)}{p} \, \mathrm{d}p \le Y_0, \\
& \int_{p^0}^{\infty} \frac{L(p)}{p^2} \, \mathrm{d}p \le X_0, \\
& X_0 P_X + Y_0 P_Y \le B, \\
& L(p) \ge 0.
\end{aligned}$$

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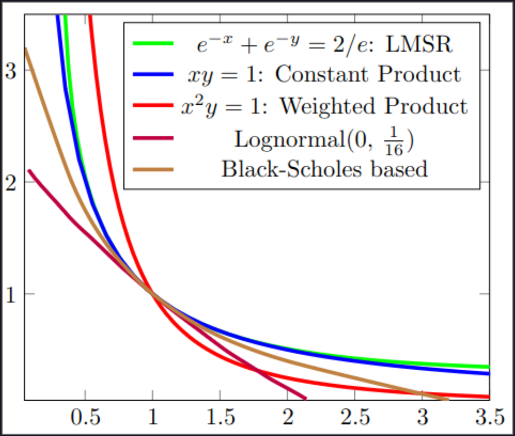
Theorem

$f(x, y) = 2 - e^{-x} - e^{-y}$ *corresponds to* $\psi(p_X, p_Y) = \frac{p_X p_Y}{(p_X + p_Y)^2}$ *on a square.*

Numerical Solutions



Conclusion

- Technique can be applied to maximize CFMM profit as well
- Constant product rule not optimal under reasonable beliefs on prices

Thank your for listening!