Statistics 225
Bayesian Statistical Analysis

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Course Overview

Prerequisites

- Probability (distributions, transformations)
- Statistical Inference (standard procedures)
- Ideally two semesters at graduate level

Broad Outline

- Univariate/multivariate models
- Hierarchical models and model checking
- Computation
- Other models (glm's, missing data, etc.)

Computing

- R - covered in class
- STAN - introduction provided

Bayesian Statistics - History

- Bayes & Laplace (late 1700s) - inverse probability
  - probability - statements about observables given assumptions about unknown parameters
  - inverse probability - statements about unknown parameters given observed data values
- Ex: given y successes in n iid trials with probability of success θ, find $\Pr(a < \theta < b|y)$
- Little progress after Bayes/Laplace except for isolated individuals (e.g., Jeffreys)
- Interest resumes in mid 1900s (the term Bayesian statistics is born)
- Computational advances in late 20th/early 21st centuries have led to increase in interest

Bayes vs Frequentist

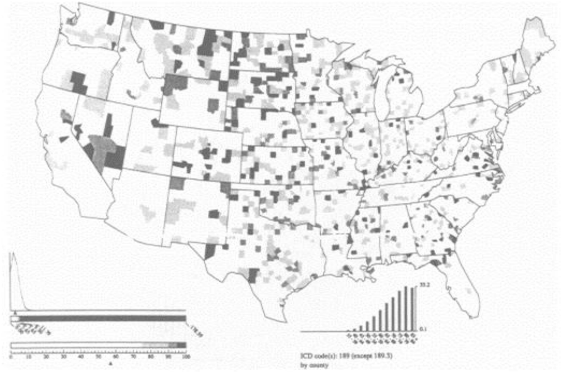
- Bayes
  - parameters as random variables
  - subjective probability (for some people)
- Frequentist
  - parameters as fixed but unknown quantities
  - probability as long-run frequency
- Some controversy in the past (and present)
- Goal here is to introduce Bayesian methods and some advantages

Some Things Not Discussed (Much)

- The following terms are sometimes associated with Bayesian statistics. They will be discussed briefly but will not receive much attention here:
  - decision theory
  - nonparametric Bayesian methods
  - subjective probability
  - objective Bayesian methods
  - maximum entropy

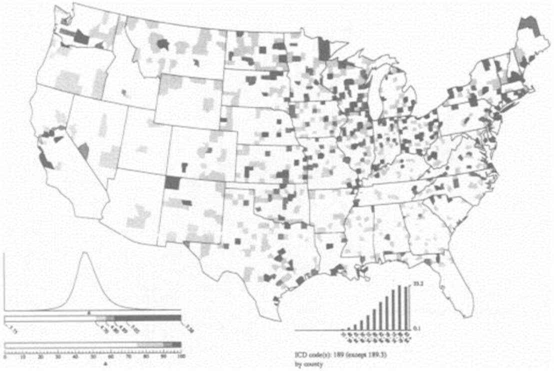
Motivating Example: Cancer Maps

- Kidney cancer mortality rates (Manton et al. - JASA, 1989)
  - Age-standardized death rates for by county



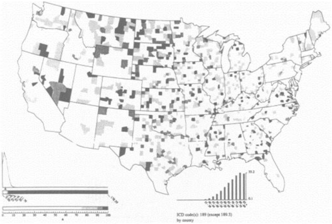
Motivating Example: Cancer Maps

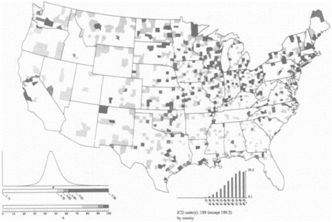
- Kidney cancer mortality rates (Manton et al. - JASA, 1989)
  - Empirical Bayes (smoothed) estimated death rates



Motivating Example: Cancer Maps

- Kidney cancer mortality rates (Manton et al. - JASA, 1989)
  - Observed (left) and Smoothed (right)





Motivating Example: SAT coaching

- SAT coaching study (Rubin - J. Educ. Stat., 1981)
  - Randomized experiments in 8 schools
  - Outcome is SAT-Verbal score
  - Effect of treatment (coaching) is estimated separately in each school using analysis of covariance

School	Estimated treatment effect	Standard error of effect estimate	Treatment effect
A	28	15	?
B	8	10	?
C	-3	16	?
D	7	11	?
E	-1	9	?
F	1	11	?
G	18	10	?
H	12	18	?

Bayesian Inference: Two key ideas

- Explicit use of probability for quantifying uncertainty
  - probability models for data given parameters
  - probability distributions for parameters
- Inference for unknowns conditional on observed data
  - inverse probability
  - Bayes' theorem (hence the modern name)
  - formal decision-making

Introduction to Bayesian Methods
Probability review

- Probability (mathematical definition):
  A set function that is
  - nonnegative
  - additive over disjoint sets
  - sums to one over entire sample space
- For Bayesian methods probability is a fundamental measure of uncertainty
  - $\Pr(1 < \bar{y} < 3|\theta = 0)$ or $\Pr(1 < \bar{y} < 3)$ is interesting before data has been collected
  - $\Pr(1 < \theta < 3|y)$ is interesting after data has been collected
- Where do probabilities come from?
  - frequency argument (e.g., 10,000 coin tosses)
  - physical argument (e.g., symmetry in coin toss)
  - subjective (e.g., if I would be willing to bet on A given 1:1 odds, then I must believe the probability of A is greater than .5)

Introduction to Bayesian Methods

Probability review

- Some terms/definitions you should know
  - joint distribution $p(u, v)$
  - marginal distribution $p(u) = \int p(u, v) dv$
  - conditional distribution $p(u|v) = p(u, v)/p(v)$
  - moments:
    $E(u) = \int u \; p(u) du = \int \int u \; p(u, v) \; dv \; du$

    $\text{Var}(u) = \int (u - E(u))^2 p(u) du$

    $E(u|v) = \int u \; p(u|v) du$ (a fn of v)

Introduction to Bayesian Methods
Probability review (cont'd)

- Some terms/definitions you should know
  - conditional distributions play a large role in Bayesian inference so the following rules are useful
    - $E(u) = E(E(u|v))$
    - $\text{Var}(u) = E(\text{Var}(u|v)) + \text{Var}(E(u|v))$
  - transformations (one-to-one)
    - denote distribution of u by $p_u(u)$
    - take $v = f(u)$
    - distribution of v is
      $p_v(v) = p_u(f^{-1}(v))$ in discrete case
      $p_v(v) = p_u(f^{-1}(v))|J|$ in continuous case
      where Jacobian J is $\left|\frac{\partial u_i}{\partial v_j}\right| = \left|\frac{\partial f^{-1}(v)}{\partial v_j}\right|$

Introduction to Bayesian Methods

Probability review - intro to simulation

- Simulation plays a big role in modern Bayesian inference and one particular transformation is important in this context
- Probability integral transform
  - suppose X is a continuous r.v. with cdf $F_X(x)$
  - then $Y = F_X(X)$ has uniform distn on 0 to 1
- Application in simulations
  - if U is uniform on $(0, 1)$ and $F(\cdot)$ is cdf of a continuous r.v.
  - then $Z = F^{-1}(U)$ is a r.v. with cdf F
  - example:
    - let $F(x) = 1 - e^{-x/\lambda}$ = exponential cdf
    - then $F^{-1}(u) = -\lambda \log(1 - u)$
    - if we have a source of uniform random numbers then we can transform to construct samples from an exponential distn
  - This is a general strategy for generating random samples

Introduction to Bayesian Methods

Notation/Terminology

- $\theta =$ unobservable quantities (parameters)
- $y =$ observed data (outcomes, responses, random variable)
- $x =$ explanatory variables (covariates, often treated as fixed)
- Don't usually distinguish between upper and lower case roman letters since everything is a random variable
- $\tilde{y} =$ unknown but potentially observable quantities (predictions, response to a different treatment)
- NOTE: don't usually distinguish between univariate, multivariate quantities

Introduction to Bayesian Methods

Notation/Terminology

- $p(\cdot)$ or $p(\cdot|\cdot)$ denote distributions (generic)
- It would take too many letters if each distribution received its own letter
- We write $Y|\mu, \sigma^2 \sim N(\mu, \sigma^2)$ to denote that Y has a normal density
- We write $p(y|\mu, \sigma^2) = N(y|\mu, \sigma^2)$ to refer to the normal density with argument y
- Same for other distributions: $\mathrm{Beta}(a, b), \mathrm{Unif}(a, b)$, $\mathrm{Exp}(\theta), \mathrm{Pois}(\lambda)$, etc.

Introduction to Bayesian Methods

The Bayesian approach

- Focus here is on three step process
  - specify a full probability model
  - posterior inference via Bayes' rule
  - model checking/sensitivity analysis
- Usually an iterative process - specify model, fit and check, then respecify model

Introduction to Bayesian Methods
Specifying a full probability model

- Data distribution $p(y|\theta) = p(\text{data} \mid \text{parameters})$
  - also known as sampling distribution
  - $p(y|\theta)$ when viewed as a function of θ is also known as the likelihood function $L(\theta|y)$
- Prior distribution $p(\theta)$
  - may contain subjective prior information
  - often chosen vague/uninformative
  - mathematical convenience
- Marginal model
  - above can be combined to determine implied marginal model for y .... $p(y) = \int p(y|\theta)p(\theta)d\theta$
  - useful for model checking
  - Bayesian way of thinking leads to new distns that can be useful even for frequentists (e.g., Beta-Binomial)

Introduction to Bayesian Methods

Posterior inference/Model checking

- Posterior inference
  - Bayes' thm to derive posterior distribution

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)} = \frac{p(y|\theta)p(\theta)}{\int p(y|\theta)p(\theta)d\theta}$$

  - probability statements about unknowns
  - formal decision-making is based on posterior distn
  - sometimes write $p(\theta|y) \propto p(\theta)p(y|\theta)$ because the denominator is a constant in terms of θ
- Model checking/sensitivity analysis
  - does the model fit
  - are conclusions sensitive to choice of prior distn/likelihood

Introduction to Bayesian Methods

Likelihood, Odds, Posteriors

- Recall that $p(\theta|y) \propto p(\theta)p(y|\theta)$
  - posterior $\propto$ prior $\times$ likelihood
  - consider two possible values of θ, say θ_1 and θ_2

$$\frac{p(\theta_1|y)}{p(\theta_2|y)} = \frac{p(\theta_1)}{p(\theta_2)} \times \frac{p(y|\theta_1)}{p(y|\theta_2)}$$

  - posterior odds $=$ prior odds $\times$ likelihood ratio
  - note likelihood ratio is still important

- Likelihood principle - if two likelihood functions agree, then the same inferences about θ should be drawn
- Traditional frequentist methods violate this
- Example: given a sequence of coin tosses with constant probability of success θ we wish to test $H_o : \theta = 0.5$
  - observe 9 heads, 3 tails in 12 coin tosses
  - if binomial sampling ($n = 12$ fixed), then
    $$L(\theta|y) = p(y|\theta) = \binom{12}{9}\theta^9(1-\theta)^3$$
    and p-value is $\Pr(y \geq 9) = .073$
  - if negative binomial sampling (sample until 3 tails), then
    $$L(\theta|y) = p(y|\theta) = \binom{11}{9}\theta^9(1-\theta)^3$$
    and p-value is $\Pr(y \geq 9) = .033$
  - but data (and likelihood function) is the same ... 9 successes, 3 failures ... and should carry the same information about θ

Introduction to Bayesian Methods
Independence

- A common statement in traditional statistics courses: assume $Y_1, \ldots, Y_n$ are iid r.v.'s
- In Bayesian class, we need to think hard about independence
- Why?
  - Consider two "indep" Bernoulli trials with probability of success θ
  - It is true that

$$p(y_1, y_2|\theta) = \theta^{y_1+y_2}(1-\theta)^{2-y_1-y_2} = p(y_1|\theta)p(y_2|\theta)$$

    so that y_1 and y_2 are independent given θ
  - But ... $p(y_1, y_2) = \int p(y_1, y_2|\theta)p(\theta)d\theta$ may not factor
  - If $p(\theta) = \mathrm{Unif}(\theta|0, 1) = 1$ for $0 < \theta < 1$, then

$$p(y_1, y_2) = \Gamma(y_1 + y_2 + 1)\Gamma(3 - y_1 - y_2)/\Gamma(4)$$

    so y_1 and y_2 are not independent in their marginal distribution

Introduction to Bayesian Methods
Exchangeability

- If independence is no longer the key concept, then what is?
- Exchangeability
  - Informal defn: subscripts don't matter
  - Formally: given events $A_1, \ldots, A_n$, we say they are exchangeable if $P(A_1 A_2 \ldots A_k) = P(A_{i_1} A_{i_2} \ldots A_{i_k})$ for every k where $i_1, i_2, \ldots, i_n$ are a permutation of the indices
  - Similarly, given random variable $Y_1, \ldots, Y_n$, we say they are exchangeable if $P(Y_1 \leq y_1, \ldots, Y_k \leq y_k) = P(Y_{i_1} \leq y_1, \ldots, Y_{i_k} \leq y_k)$ for every k

Introduction to Bayesian Methods

Exchangeability and independence

- Relationship between exchangeability and independence
  - r.v.'s that are iid given θ are exchangeable
  - an infinite sequence of exchangeable r.v.'s can always be thought of as iid given some parameter (de Finetti)
  - note previous point requires an infinite sequence
- What is not exchangeable?
  - time series, spatial data
  - may become exchangeable if we explicitly include time or spatial location in the analysis
  - i.e., $y_1, y_2, \ldots, y_t, \ldots$ are not exchangeable but $(t_1, y_1), (t_2, y_2), \ldots$ may be

Introduction to Bayesian Methods

A simple example

- Hemophilia - blood clotting disease
  - sex-linked genetic disease on X chromosome
  - males (XY) - affected or not
  - females (XX) - may have 0 copies of disease gene (not affected), 1 copy (carrier), 2 copies (usually fatal)
- Consider a woman – brother is a hemophiliac, father is not
  - we ignore the possibility of a mutation introducing the disease
  - woman's mother must be a carrier
  - woman inherits one X from mother
    $-->$ 50/50 chance of being a carrier
- Let $\theta = 1$ if woman is carrier, 0 if not
  - a priori we have $\Pr(\theta = 1) = \Pr(\theta = 0) = 0.5$
- Let y_i = status of woman's ith male child
  (1 if affected, 0 if not)

Introduction to Bayesian Methods

A simple example (cont'd)

- Given two unaffected sons (not twins), what inference can be drawn about θ?
- Assume two sons are iid given θ
- $\Pr(y_1 = y_2 = 0|\theta = 1) = 0.5 * 0.5 = .25$
  $\Pr(y_1 = y_2 = 0|\theta = 0) = 1 * 1 = 1.00$
- Posterior distn by Bayes' theorem

$$\begin{aligned}
\Pr(\theta = 1|y) &= \frac{\Pr(y|\theta = 1)\Pr(\theta = 1)}{\Pr(y)} \\
&= \frac{\Pr(y|\theta = 1)\Pr(\theta = 1)}{\Pr(y|\theta = 1)\Pr(\theta = 1) + \Pr(y|\theta = 0)\Pr(\theta = 0)} \\
&= \frac{.25 * .5}{.25 * .5 + 1 * .5} = .2
\end{aligned}$$

Introduction to Bayesian Methods
A simple example (cont'd)

- Odds version of Bayes' rule
  - prior odds $\Pr(\theta = 1)/\Pr(\theta = 0) = 1$
  - likelihood ratio $\Pr(y|\theta = 1)/\Pr(y|\theta = 0) = 1/4$
  - posterior odds $= 1/4$
    (posterior prob $= .25/(1 + .25) = .20$)
- Updating for new information
  - suppose that a 3rd son is born (unaffected)
  - note: if we observe an affected child, then we know $\theta = 1$ since that outcome is assumed impossible when $\theta = 0$
  - two approaches to updating analysis
    - redo entire analysis (y_1, y_2, y_3 as data)
    - update using only new data (y_3)

Introduction to Bayesian Methods
A simple example (cont'd)

- Updating for new information - redo analysis
  - as before but now $y = (0, 0, 0)$
  - $\Pr(y|\theta = 1) = .5 * .5 * .5 = .125$,
    $\Pr(y|\theta = 0) = 1$
  - $\Pr(\theta = 1|y) = .125 * .5/(.125 * .5 + 1 * .5) = .111$
- Updating for new information - updating
  - take previous posterior distn as new prior distn
    ($\Pr(\theta = 1) = .2$ and $\Pr(\theta = 0) = .8$)
  - take data as consisting only of y_3
  - $\Pr(\theta = 1|y_3) = .5 * .2/(.5 * .2 + 1 * .8) = .111$
  - same answer!

Single Parameter Models
Introduction

- We introduce important concepts/computations in the one-parameter case
- There is generally little advantage to the Bayesian approach in these cases
- The benefits of the Bayesian approach are more obvious in hierarchical (often random effects) models
- Main approach is to teach via example
- First example is binomial data (Bernoulli trials)
  - easy
  - historical interest (Bayes, Laplace)
  - representative of a large class of distns (exponential families)

Single Parameter Models
Binomial Model

- Consider n exchangeable trials
- Data can be summarized by total # of successes
- Natural model: define θ as probability of success and take $Y \sim \text{Bin}(n, \theta)$

$$p(y|\theta) = \text{Bin}(y|n, \theta) = \binom{n}{y} \theta^y (1-\theta)^{n-y}$$

- Question - do we have to be explicit about conditioning on n? (usually are not)
- Prior distribution: To start assume $p(\theta) = \text{Unif}(\theta|0, 1)$

Single Parameter Models

Binomial Model

- Posterior distribution:

$$
\begin{aligned}
p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \Big/ \int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}d\theta \\
&= (n+1)\binom{n}{y}\theta^y(1-\theta)^{n-y} = \frac{(n+1)!}{y!(n-y)!}\theta^y(1-\theta)^{n-y} \\
&= \frac{\Gamma(n+2)}{\Gamma(y+1)\Gamma(n-y+1)}\theta^{y+1-1}(1-\theta)^{n-y+1-1} \\
&= \text{Beta}(y+1, n-y+1)
\end{aligned}
$$

- Note: could have noticed $p(\theta|y) \propto \theta^y(1-\theta)^{n-y}$ and inferred it is a $\text{Beta}(y+1, n-y+1)$ distribution (formal calculation confirms this)

Single Parameter Models
Binomial Model

- Inferences from the posterior distribution
  - point estimation
    - posterior mean $= (y+1)/(n+2)$
      (compromise between sample proportion $\frac{y}{n}$ and prior mean $\frac{1}{2}$)
    - posterior mode $= y/n$
    - best point estimate depends on loss function
    - posterior variance $= \left(\frac{y+1}{n+2}\right)\left(\frac{n-y+1}{n+2}\right)\left(\frac{1}{n+3}\right)$
  - interval estimation
    - 95% central posterior interval - find a,b s.t.
      $\int_0^a \text{Beta}(\theta|y+1, n-y+1)d\theta = .025$ and
      $\int_0^b \text{Beta}(\theta|y+1, n-y+1)d\theta = .975$
    - alternative is highest posterior density region
    - note this interval has the interpretation we want to give to traditional CIs
  - hypothesis test – don't say anything about this now

Single Parameter Models

Binomial Model

- Inference by simulation
  - the inferences mentioned (point estimation, interval estimation) can be done via simulation
  - simulate 1000 draws from the posterior distribution
    - available in standard packages
    - we will discuss algorithms for harder problems later
  - point estimates easy to compute (now include Monte Carlo error)
  - interval estimates easy – find percentiles of the simulated values

Single Parameter Models

Prior distributions

- Where do prior distributions come from?
  - a priori knowledge about θ ("thinking deeply about context")
  - population interpretation (a population of possible θ values)
  - mathematical convenience
- Frequently rely on asymptotic results (to come) which guarantee that likelihood will dominate the prior distn in large samples

Single Parameter Models

Conjugate prior distributions

- Consider Beta(α, β) prior distn for binomial model
  - think of α, β as fixed now (but these could also be random and given their own prior distn)
  - $$\begin{aligned} p(\theta|y) &\propto \theta^y(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &\propto \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \end{aligned}$$
  - recognize as kernel of Beta$(y+\alpha, n-y+\beta)$
  - example of conjugate prior distn - posterior distn is in the same parametric family as the prior distn
  - convenient mathematically
  - convenient interpretation - prior in this case is like observing α successes in $\alpha+\beta$ "prior" trials

Single Parameter Models

Conjugate prior distributions - general

- Definition:
  Let F be a class of sampling distn ($p(y|\theta)$).
  Let P be a class of prior distns ($p(\theta)$).
  P is **conjugate** for F if $p(\theta) \in P$ and $p(y|\theta) \in F$ implies that $p(\theta|y) \in P$
- Not a great definition ... trivially satisfied by $P = \{$ all distns$\}$ but this is not an interesting case
- Exponential families (most common distns):
  the only distns that are finitely parametrizable
  and have conjugate prior families

Single Parameter Models

Conjugate prior distributions - exponential families

- The density of an exponential family can be written as

$$p(y_i|\theta) = f(y_i)g(\theta)e^{\phi(\theta)^t u(y_i)}$$

$$p(y_1, \ldots, y_n|\theta) = (\prod_{i=1}^{n} f(y_i))g(\theta)^n e^{\phi(\theta)^t t(y)}$$

  with $\phi(\theta)$ denoting the natural parameter(s) and $t(y) = \sum_i u(y_i)$ denoting the sufficient statistic(s)
- Note that $p(\theta) \propto g(\theta)^\eta e^{\phi(\theta)^t \nu}$ will be conjugate family
- Binomial example
  - $p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$
  - exponential family with $\phi(\theta) = \log(\theta/(1-\theta))$ and $g(\theta) = 1 - \theta$
  - conjugate prior distn is $\theta^\nu(1-\theta)^{\eta-\nu}$ (Beta distribution)

Single Parameter Models

Conjugate prior distributions - normal distn with known variance

- Normal example
  - $p(y_i|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(y_i-\theta)^2/2\sigma^2}$
  - exponential family with $\phi(\theta) = \theta/\sigma$ and $g(\theta) = e^{-\theta^2/2\sigma^2}$
  - conjugate prior distn is exponential of quadratic form in θ (i.e., normal distribution)
  - take prior distn as $\theta \sim N(\mu, \tau^2)$
  - posterior distn is $p(\theta|y) = N(\theta|\hat{\mu}, V)$ with

$$\hat{\mu} = \frac{\frac{n}{\sigma^2}\bar{y} + \frac{1}{\tau^2}\mu}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \text{ and } V = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

Single Parameter Models

Conjugate prior distributions - normal distn with known variance

- Normal example (cont'd)
  - posterior distribution is $p(\theta|y) = N(\theta|\hat{\mu}, V)$ with

$$\hat{\mu} = \frac{\frac{n}{\sigma^2}\bar{y} + \frac{1}{\tau^2}\mu}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \text{ and } V = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

  - posterior mean $=$ wtd average of prior mean and sample mean
  - weights depend on precision (inverse variance) of the prior distribution and the data distribution
  - posterior precision is the sum of the prior precision and the data precision
  - if $n \to \infty$ then posterior distn resembles $p(\theta|y) = N(\theta|\bar{y}, \sigma^2/n)$; like classical sampling distn result (so the data dominates the prior distn for large n)

Single Parameter Models

Conjugate prior distributions - general

- Advantages
  - mathematically convenient
  - easy to interpret
  - can provide good approx to many prior opinions (especially if we allow mixtures of distns from the conjugate family)
- Disadvantages
  - may not be realistic

Single Parameter Models

Nonconjugate prior distributions

- No real difference conceptually in how analysis proceeds
- Harder computationally
- One simple idea is grid-based simulation
  - specify prior distn on a grid $\Pr(\theta = \theta_i) = \pi_i$
  - compute likelihood on same grid $l_i = p(y|\theta_i)$
  - posterior distn lives on the grid with $\Pr(\theta = \theta_i|y) = \pi_i^* = \pi_i l_i / (\sum_j \pi_j l_j)$
  - can sample from this posterior distn easily in R
  - can do better with a trapezoidal approx to the prior distn
- However there are serious problems with grid-based simulation
- We will see better computational approaches

Single Parameter Models

Noninformative prior distributions

- Sometimes there is a desire to have the prior distn play a minimal role in forming the posterior distn (why?)
- To see how this might work recall our normal example with $y_1, \ldots, y_n|\theta \sim \text{iid} N(\theta, \sigma^2)$ and $p(\theta|\mu, \tau^2) = N(\theta|\mu, \tau^2)$ where σ^2, μ, τ^2 are known
  - a conjugate family with $p(\theta|y) = N(\theta|\hat{\mu}, V)$ where

$$\hat{\mu} = \frac{\frac{n}{\sigma^2}\bar{y} + \frac{1}{\tau^2}\mu}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \text{ and } V = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

  - if $\tau^2 \to \infty$, then $p(\theta|y) \approx N(\theta|\bar{y}, \sigma^2/n)$
    (this yields the same estimates and intervals as classical methods; can be thought of as non-informative)
  - same result would be obtained by taking $p(\theta) \propto 1$
    BUT that is not a proper prior distn
  - we can use an improper prior distn but must
    check that the posterior distn is a proper distn

Single Parameter Models

Noninformative prior distributions

- How do we find noninformative prior distributions?
- Flat or uniform distributions
  - did the job in the binomial and normal cases
  - makes each value of θ equally likely
  - but on what scale (should every value of $\log \theta$ be equally likely or every value of θ)
- Jeffrey's prior
  - invariance principle – a rule for creating noninformative prior distns should be invariant to transformation
  - this means that if $p_\theta(\theta)$ is prior distn for θ and we consider $\phi = h(\theta)$, then our rule should create $p_\phi(\phi) = p_\theta(h^{-1}(\phi)) \, |d\theta/d\phi|$
  - Jeffrey's suggestion to use $p(\theta) \propto J(\theta)^{1/2}$ where $J(\theta)$ is the Fisher information satisfies this principle
  - gives flat prior for θ in normal case
  - does this work for multiparameter problems?

Single Parameter Models

Noninformative prior distributions

- How do we find noninformative prior distributions? (cont'd)
- Pivotal quantities
  - location family has $p(y - \theta|\theta) = f(y - \theta)$ so should expect $p(y - \theta|y) = f(y - \theta)$ as well ...... this suggests $p(\theta) \propto 1$
  - similar argument for scale family suggests $p(\theta) \propto 1/\theta$ (where θ is a scale parameter like normal s.d.)
- Vague, diffuse distributions
  - use conjugate or other prior distn with large variance

Single Parameter Models
Noninformative prior distributions - example

- Binomial case
  - Uniform on θ is Beta$(1, 1)$
  - Jeffreys' prior distn is Beta$(1/2, 1/2)$
  - Uniform on natural parameter $\log(\theta/(1-\theta))$ is Beta$(0, 0)$ (an improper prior distn)
- Summary on noninformative distn
  - very difficult to make this idea rigorous since it requires a definition of "information'
  - can be useful as a first approximation or first attempt
  - dangerous if applied automatically without thought
  - improper distributions can cause serious problems (improper posterior distns) that are hard to detect
  - some prefer vague, diffuse, or "weakly informative" proper distributions as a way of expressing ignorance

Single Parameter Models

Weakly informative prior distributions

- Proper distributions
- Intentionally made weaker (more diffuse) than the actual prior information that is available
- Example 1 - normal mean
  - Can take the prior distribution to be $N(0, A^2)$ where A is chosen based on problem context ($2A$ is a plausible upper bound on θ)
- Example 2 - binomial proportion
  - Can take the prior distribution to be $N(0.5, A^2)$ where A is chosen so that $0.5 \pm 2A$ contains all plausible values of θ

Multiparameter Models

Introduction

- Now write $\theta = (\theta_1, \theta_2)$ (at least two parameters)
- θ_1 and θ_2 may be vectors as well
- Key point here is how the Bayesian approach handles "nuisance" parameters
- Posterior distn $p(\theta_1, \theta_2 | y) \propto p(y | \theta_1, \theta_2) p(\theta_1, \theta_2)$
- Suppose θ_1 is of primary interest, i.e., want $p(\theta_1 | y)$
  - $p(\theta_1 | y) = \int p(\theta_1, \theta_2 | y) d\theta_2$ analytically or by numerical integration
  - $p(\theta_1 | y) = \int p(\theta_1 | \theta_2, y) p(\theta_2 | y) d\theta_2$ (often a convenient way to calculate)
  - $p(\theta_1 | y) = \int p(\theta_1, \theta_2 | y) d\theta_2$ by simulation (generate simulations of both and toss out the θ_2's)
- Note: Bayesian results still usually match those of traditional methods. We don't see differences until hierarchical models

Multiparameters Models

Normal example

- $y_1, y_2, \ldots, y_n | \mu, \sigma^2$ are iid $N(\mu, \sigma^2)$
- Prior distn: $p(\mu, \sigma^2) \propto 1/\sigma^2$
  - indep non-informative prior distns for μ and σ^2
  - equivalent to $p(\mu, \log \sigma) \propto 1$
  - not a proper distn
- Posterior distn:

$$
\begin{aligned}
p(\mu, \sigma^2 | y) &\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1} \exp[-\frac{1}{2\sigma^2} \sum_i (y_i - \mu)^2] \\
&\propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_i (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right)\right]
\end{aligned}
$$

  - note that μ, σ^2 are not indep in their posterior distn
  - posterior distn depends on data only through the sufficient statistics

Multiparameters Models
Normal example (cont'd)

- Further examination of joint posterior distribution

$$p(\mu, \sigma^2|y) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_i (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right)\right]$$

  - conditional posterior distn $p(\mu|\sigma^2, y)$
    - examine joint posterior distn but now think of σ^2 as known
    - focus only on μ terms
    - $p(\mu|\sigma^2, y) \propto \exp[-\frac{1}{2\sigma^2} n(\bar{y} - \mu)^2]$
    - just like known variance case
    - recognize $\mu|\sigma^2, y \sim N(\bar{y}, \sigma^2/n)$
  - marginal posterior distn of σ^2, i.e., $p(\sigma^2|y)$
    - $p(\sigma^2|y) = \int p(\mu, \sigma^2|y) d\mu$
    - $p(\sigma^2|y) \propto (\sigma^2)^{-(n+1)/2} \exp[-\frac{1}{2\sigma^2} \sum_i (y_i - \bar{y})^2]$
    - known as scaled-inverse-$\chi^2(n-1, s^2)$ distn with $s^2 = \sum_i (y_i - \bar{y})^2/(n-1)$

Multiparameters Models

Normal example (cont'd)

- Recall joint posterior distribution

$$p(\mu, \sigma^2|y) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_i (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right)\right]$$

- A useful identity for deriving marginal distributions from the joint distribution and a conditional distribution
  - marginal posterior distn of σ^2 is defined as $p(\sigma^2|y) = \int p(\mu, \sigma^2|y) d\mu$
  - note also that $p(\sigma^2|y) = p(\mu, \sigma^2|y)/p(\mu|\sigma^2, y)$
  - LHS doesn't have μ, RHS does
  - equality must be true for any choice of μ
  - evaluate this ratio at $\mu = \bar{y}$ (why? the conditional density is $N(\mu|\bar{y}, \sigma^2/n)$)
  - this also yields $p(\sigma^2|y) \propto (\sigma^2)^{-(n+1)/2} \exp[-\frac{1}{2\sigma^2} \sum_i (y_i - \bar{y})^2]$

Multiparameters Models
Normal example (cont'd)

- Further examination of joint posterior distribution

$$p(\mu,\sigma^2|y) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_i (y_i-\bar{y})^2 + n(\bar{y}-\mu)^2\right)\right]$$

  - so far, $p(\mu,\sigma^2|y) = p(\sigma^2|y)p(\mu|\sigma^2,y)$
  - this factorization can be used to simulate from joint posterior distn
    - generate σ^2 from Inv-$\chi^2(n-1,s^2)$ distn
    - then generate μ from $N(\bar{y},\sigma^2/n)$ distn
  - often most interested in $p(\mu|y)$
    - $p(\mu|y) = \int_0^\infty p(\mu,\sigma^2|y)d\sigma^2 \propto \left[1+\frac{n(\mu-\bar{y})}{(n-1)s^2}\right]^{-n/2}$
    - $\mu|y \sim t_{n-1}(\bar{y},s^2/n)$ (a t-distn)
    - recall traditional result $\frac{\bar{y}-\mu}{s/\sqrt{n}}|\mu,\sigma^2 \sim t_{n-1}$
      (note result doesn't depend at all on σ^2)

Multiparameters Models
Normal example (cont'd)

- Further examination of joint posterior distribution

$$p(\mu, \sigma^2|y) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\left(\sum_i (y_i - \bar{y})^2 + n(\bar{y} - \mu)^2\right)\right]$$

  - consider $\tilde{y}$ a future draw from the same population
  - what is the predictive distn of $\tilde{y}$, i.e., $p(\tilde{y}|y)$
  - $p(\tilde{y}|y) = \int\int p(\tilde{y}|\mu, \sigma^2, y)p(\mu, \sigma^2|y)d\mu\ d\sigma^2$
  - note first term in integral doesn't depend on y .... given params we know distn of $\tilde{y}$ is $N(\mu, \sigma^2)$
  - predictive distn by simulation
    (simulate $\sigma^2 \sim \text{Inv-}\chi^2(n-1, s^2)$,
    then $\mu \sim N(\bar{y}, \sigma^2/n)$, then $\tilde{y} \sim N(\mu, \sigma^2)$)
  - predictive distn analytically (can proceed as for μ by first conditioning on σ^2)
    $\tilde{y}|y \sim t_{n-1}(\bar{y}, (1 + \frac{1}{n})s^2)$

Multiparameters Models

Normal example - conjugate prior distn

- It can be hard to find conjugate prior distributions for multiparameter problems
- It is possible for the normal (two-parameter) example
- Conjugate prior distribution is product of $\sigma^2 \sim \text{Inv-}\chi^2(\nu_o, \sigma_o^2)$ and $\mu|\sigma^2 \sim N(\mu_o, \sigma^2/\kappa_o)$
- Conditional distribution for μ is equivalent to κ_o observations on the scale of y
- This is known as the Normal-Inv$\chi^2(\mu_o, \kappa_o; \nu_o, \sigma_o^2)$ prior
- The posterior distribution is of the same form with
    - $\mu_n = \frac{\kappa_o}{\kappa_o+n}\mu_o + \frac{n}{\kappa_o+n}\bar{y}$
    - $\kappa_n = \kappa_o + n$
    - $\nu_n = \nu_o + n$
    - $\nu_n\sigma_n^2 = \nu_o\sigma_o^2 + (n-1)s^2 + \frac{\kappa_o n}{\kappa_o+n}(\bar{y} - \mu_o)^2$

Multiparameters Models

Normal example - other prior distns (cont'd)

- Semi-conjugate analysis
  - for conjugate distn, the prior distn for μ depends on scale parameter σ (unknown)
  - may want to allow info about μ that does not depend on σ
  - consider independent prior distributions $\sigma^2 \sim \text{Inv-}\chi^2(\nu_o, \sigma_o^2)$ and $\mu \sim N(\mu_o, \tau_o^2)$
  - may call this semi-conjugate
  - note that given σ^2, analysis for μ is conjugate normal-normal case so that $\mu|\sigma^2, y \sim N(\mu_n, \tau_n^2)$ with

$$\mu_n = \frac{\frac{1}{\tau_o^2}\mu_o + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_o^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \tau_n^2 = \frac{1}{\frac{1}{\tau_o^2} + \frac{n}{\sigma^2}}$$

Multiparameters Models

Normal example - other prior distns (cont'd)

- Semi-conjugate analysis (cont'd)
  - $p(\sigma^2|y)$ is not recognizable distn
    - calculate as $p(\sigma^2|y) = \int \prod_{i=1}^{n} N(y_i|\mu,\sigma^2)N(\mu|\mu_o,\tau_o^2)\text{Inv}-\chi^2(\sigma^2|\nu_o,\sigma_o^2)d\mu$
    - or calc $p(\sigma^2|y) = p(\mu,\sigma^2|y)/p(\mu|\sigma^2,y)$ (RHS evaluated at convenient choice of μ)
    - use a 1-dimensional grid approximation or some other simulation technique
- Multivariate normal case
  - no details here (see book)
  - discussion is almost identical to that for univariate normal distn with Inv-Wishart distn in place of the Inv-χ^2

Multiparameters Models

Multinomial data

- Data distribution

$$p(y|\theta) = \prod_{j=1}^{k} \theta_j^{y_j}$$

  where $\theta =$ vector of probabilities with $\sum_{j=1}^{k} \theta_j = 1$
  and $y =$ vector of counts with $\sum_{j=1}^{k} y_j = n$
- Conjugate prior distn is the Dirichlet(α) distn ($\alpha > 0$) (multivariate generalization of the beta distn)

$$p(\theta) = \prod_{j=1}^{k} \theta_j^{\alpha_j - 1}$$

  for vectors θ such that $\sum_{j=1}^{k} \theta_j = 1$
  - $\alpha = 1$ yields uniform prior distn on θ vectors (noninformative? ... favors uniform distn)
  - $\alpha = 0$ uniform on $\log \theta$ (noninformative but improper)
- Posterior distn is Dirchlet($\alpha + y$)

Multiparameters Models

A non-standard example: logistic regression

- A toxicology study (Racine et al, 1986, Applied Statistics)
- $x_i = \log(\text{dose}), i = 1, \ldots, k$ (k dose levels)
- $n_i =$ animals given ith dose level
- $y_i =$ number of deaths
- Goals:
  - traditional inference for parameters α, β
  - special interest in inference for LD50 (dose at which expect 50% would die)

Multiparameters Models
Logistic regression (cont'd)

- Data model specification
  - within group (dose): exchangeable animals so model $y_i|\theta_i \sim \text{Bin}(n_i, \theta_i)$
  - between groups: non-exchangeable (higher dose means more deaths); many possible models including

$$\text{logit}(\theta_i) = \log\left(\frac{\theta_i}{1-\theta_i}\right) = \alpha + \beta x_i$$

  - resulting data model

$$p(y|\alpha,\beta) = \prod_{i=1}^{k}\left(\frac{e^{\alpha+\beta x_i}}{1+e^{\alpha+\beta x_i}}\right)^{y_i}\left(\frac{1}{1+e^{\alpha+\beta x_i}}\right)^{n_i-y_i}$$

- Prior distn
  - noninformative: $p(\alpha,\beta) \propto 1$ ... is posterior distn proper?
  - answer is yes but it is not-trivial to show
  - should we restrict $\beta > 0$??

Multiparameters Models

Logistic regression example (cont'd)

- Posterior distn: $p(\alpha, \beta|y) \propto p(y|\alpha, \beta)p(\alpha, \beta)$

$$p(\alpha, \beta|y) = \prod_{i=1}^{k} \left(\frac{e^{\alpha+\beta x_i}}{1 + e^{\alpha+\beta x_i}} \right)^{y_i} \left(\frac{1}{1 + e^{\alpha+\beta x_i}} \right)^{n_i - y_i}$$

- Grid approximation
  - obtain crude estimates of α, β
    (perhaps by standard logistic regression)
  - define grid centered on crude estimates
  - evaluate posterior density on 2-dimensional grid
  - sample from discrete approximation
  - refine grid and repeat if necessary
- Grid approximations are risky because they may miss important parts of the distn
- More sophisticated approaches will be developed later (MCMC)

Multiparameters Models
Logistic regression example (cont'd)

- Inference for LD50
  - want x_i such that $\theta_i = 0.5$
  - turns out $x_i = -\alpha/\beta$
  - with simulation it is trivial to get posterior distn of $-\alpha/\beta$
  - note that using MLEs it would be easy to get estimate but hard to get standard error
  - doesn't make sense to talk about LD50 if $\beta < 0$ .... could do inference in two steps
    - $\Pr(\beta > 0)$
    - distn of LD50 given $\beta > 0$
- Real-data example (handout)

Large Sample Inference

Asymptotics in Bayesian Inference

- "Optional" because Bayesian methods provide proper finite sample inference, i.e. we have a posterior distribution for θ that is valid regardless of sample size
- Large sample results are still interesting – Why?
  - theoretical results (the likelihood dominates the prior so that frequentist asymptotic results apply to Bayesian methods also)
  - approximation to the posterior distn
  - normal approx can provide useful information to check simulations from actual posterior distn

Large Sample Inference

Asymptotics in Bayesian Inference

- Large sample results are still interesting - Why? (continuation)
  - approximation to the posterior distn
    - normal approx is easy (need only posterior mean and s.d.).
    - normal approx often adequate if few dimensions (especially after transforming)
  - normal theory helps interprete posterior pdf's: for d-dimension normal approx
    - $-2\log(\text{density}) = (x-\mu)'\Sigma^{-1}(x-\mu)$ is approximately χ^2_d as $n \to \infty$
    - 95% posterior confidence region for μ contains all μ with posterior density $\geq \exp\{-0.5\chi^2_{d,0.95}\} \times \max p(\theta|y)$

Large Sample Inference
Consistency

- Let $f(y)$ be true data generating distn
- Let $p(y|\theta)$ be the model being fit
- Finite parameter space Θ.
  - true value generating the data is $\theta_0 \in \Theta$ (i.e. $f(y) = p(y|\theta_o)$)
  - assume $p(\theta_0) > 0$.

  then

  $$p(\theta = \theta_0|y) \to 1 \text{ as } n \to \infty$$

- Same result if $p(y|\theta)$ is not the right family of distn by taking θ_0 to be the Kullback-Leibler minimizer, i.e., θ_0 s.t. $H(\theta) = \int f(y) \log\left(\frac{f(y)}{p(y|\theta)}\right) dy$ is minimized
- Can extend to more general parameter spaces

Large Sample Inference

Asymptotic Normality

(1-dimension parameter space)

Theorem (BDA3, pg 587)

Under some regularity conditions (notably that θ_0 not be on the boundary of Θ), as $n \to \infty$, the posterior distribution of θ approaches normality with mean θ_0 and variance $(nJ(\theta_0))^{-1}$, where θ_0 is the true value or the value that minimizes the Kullback-Leibler information and $J(\cdot)$ is the Fisher information.

Large Sample Inference

Asymptotic Normality

- Problems that affect Bayesian and classical arguments
  - If "true" θ_0 is on the boundary of the parameter space, then no asymptotic normality
  - Sometimes the likelihood is unbounded

    e.g.

$$f(y|\lambda,\mu_1,\sigma_1,\mu_2,\sigma_2)=\lambda f_1(y|\theta)+(1-\lambda)f_2(y|\theta)$$

where

$$f_i(y|\theta)=\frac{1}{\sqrt{2\pi}\sigma_i}e^{-\frac{1}{2}\left(\frac{Y-\mu_i}{\sigma_i}\right)^2}\qquad i=1,2$$

If we take $\mu_1=y_1$ and $\sigma_1\to 0$, then $f(\theta|y)$ is unbounded

Large Sample Inference
Asymptotic Normality

- Problems that only affect Bayesians
  - improper posterior distns (already discussed)
  - prior distn that excludes "true" θ_0
  - problems where the number of parameters increase with the sample size, e.g.,

$$\begin{array}{c} Y_i|\theta_i \sim N(\theta_i, 1) \\ \theta_i|\mu, \tau^2 \sim N(\mu, \tau^2) \end{array} \qquad i = 1, \ldots, n$$

    then asymptotic results hold for μ, τ^2 but not θ_i

Large Sample Inference
Asymptotic Normality

- Problems that only affect Bayesians (cont'd)
  - parameters not identified.
    e.g.
    $$\begin{pmatrix} U \\ V \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right]$$
    if you observe only U or V for each pair, there is no information about ρ.
  - tails of the distribution may not be normal, e.g., our logistic regression example