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Time Hierarchy Theorem:

For every proper $f(n) \geq n$ $TIME(f(n)) \subsetneq TIME(f(2n)^3)$.

Space Hierarchy Theorem

For every proper function $f(n) \geq \log n$,
$SPACE(f(n)) \subsetneq SPACE(f(n) \cdot \log(f(n)))$

(Proof Similar to the Time Hierarchy Theorem).

Here are some of the common complexity classes that we will study:

$$\mathcal{L} = SPACE(\log n)$$

input read only tape
log n cells on the R/W work tape

$$PSPACE = \bigcup_{k \geq 1} SPACE(n^k)$$

$$P = \bigcup_{k \geq 1} TIME(n^k)$$

$$EXP = \bigcup_{k \geq 1} TIME(2^{n^k})$$

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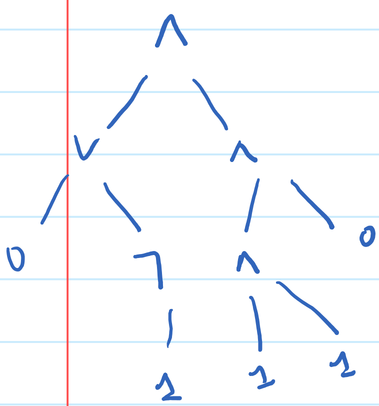
Sample Problems for each Complexity class:

L : FVAL (function evaluation)



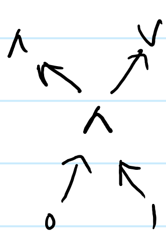
Input: Boolean formula (each variable only appears once).
+ S truth assignment for the vars.
Does the function evaluate to true?

PSPACE : 2-person game
ex. Generalized Geography
Input: Directed Graph.
Players take turns selecting an outgoing edge from the current node. Current node is updated after following the selected edge.
Player loses if no out-going edge.
Does Player 1 have a winning strategy?

P CVAL, linear programming, max flow...

→ Given a Boolean circuit + an assignment to the inputs, what are the values of the outputs?

Difference between a circuit + a formula is the circuit can have fan out > 1:



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What are the relationships between these classes?

$$P \subseteq TIME(2^n) \underset{\text{Time Hierarchy.}}{\subsetneq} TIME\left(2^{(2n)\cdot 3}\right) \subseteq EXP$$

$$\Rightarrow P \subsetneq EXP$$

$$\mathcal{L} = SPACE(\log n) \underset{\text{S.H.}}{\subsetneq} SPACE(\log^2 n) \subseteq PSPACE$$

$$\mathcal{L} \subsetneq PSPACE$$

Next we shall see: $\mathcal{L} \subseteq P$ → conjectured to be not equal but unknown.

PSPACE $\subseteq$ EXP

The entire configuration of a TM can be described by: contents of work tape.
location of head(s).
State.

⇒ String: $\sigma_1 \sigma_2 \cdots \sigma_{i-1} \overset{\text{state}}{q} \sigma_i \cdots \sigma_m$ ← location of head.

C_{start}: $q_{start} \underbrace{x_1 x_2 \cdots x_n}_{\text{input.}}$

Easy to determine if $C \rightarrow C'$ in one step.

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For TM M & input x, define Configuration graph:

nodes $\longleftrightarrow$ set of all configurations.
Edge (c, c') if c yields c' in one step.
(out-degree ≤ 1).

The # of Configurations of a 2-tape TM:
$\Rightarrow$ Tape 1: Read-only input.
$\Rightarrow$ Tape 2: R/W work tape w/ $\leq t(n)$ cells

$$\Rightarrow \underbrace{n \times t(n)}_{\text{\# head positions}} \times |Q| \times |\Sigma|^{t(n)}$$

for $t(n) = \log n$
configs is $\leq n^k$

Alg: Calculate the # of Configurations (or any poly u.b.)
Start w/ C_{start} & follow the path of configurations.
Stop if # steps $\geq$ # configurations or if halting configuration is reached.

don't actually need to build the whole graph.

$t = O(\log n)$.

$\Rightarrow L \subseteq P$. $n \times \log n \times c \times (c_1)^{c_2 \log n} \leq n^k$.

If $t(n) = n^c$ # configs $\leq n \times n^c \times c_0 \times c_1^{n^c} \leq 2^{n^k}$

c_0: # states

c_1: # chars in Σ

for some k and $n \geq n_0$

configurations is exp in n.
Time is now: 2^{n^k}.

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So we have $\mathcal{L} \subseteq P \subseteq PSPACE \subseteq EXP$

We know $\mathcal{L} \subsetneq PSPACE$

but we don't know if $\mathcal{L} \subsetneq P$ or $P \subsetneq PSPACE$ or both.

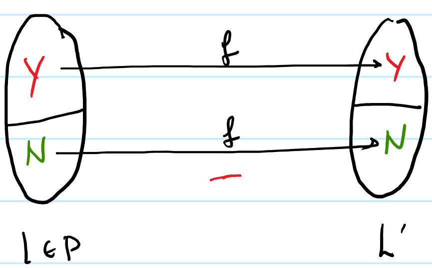
(at least one inequality must hold).

Conjecture is that all containments are strict.

We don't know if $\mathcal{L} \subsetneq P$ but we can identify a problem in P that is least likely to be in $\mathcal{L}$.

Want a P-complete problem: L'

$\forall\, L \in P \quad L \propto L'$ so that if $L' \in \mathcal{L}$ then $\forall\, L \in P \rightarrow L \in \mathcal{L}$.



To compute L on input x:
Compute $f(x)$ then run log space alg for L' on $f(x)$.

Need log space reductions!

Not obvious how to compose log space reductions.

read-only input
write-only output
log space work tape

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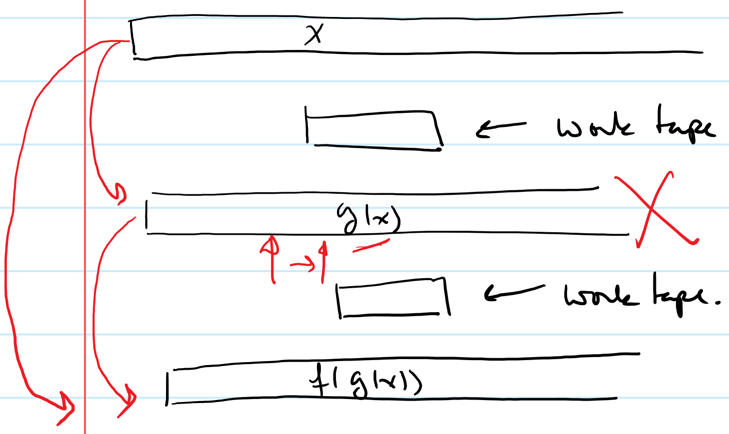


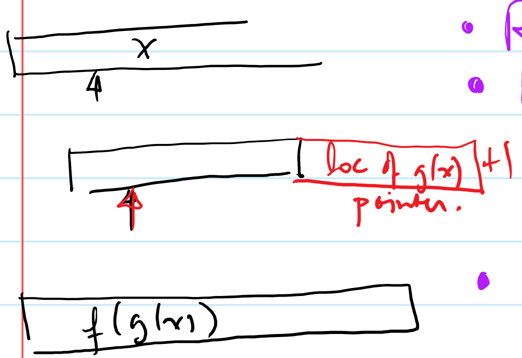
Want: log-space computation that computes $x \to f(g(x))$ in log space

Can not write $g(x)$ in its entirety.

Simulate M_f: keep a register with location of read-head $O(\log n)$ bits.

When simulation needs to read i^{th} location of $f(x)$

- Remember current state $O(\log |Q_{M_f}|)$ bits.
- Restart M_g until i^{th} location of $f(x)$ is written. Don't write the other outputs chars of $f(x)$.
- Simulate next step of M_f.



Space: work tape for M_g. –
work tape for M_f. –
State of M_f. –
Counter for i $\Rightarrow O(\log n)$.

$L_1 \in P$ $\quad L_1 \propto L_2$ — by logspace reduction.

If $L_2 \in \mathcal{L}$ then $L_1 \propto \mathcal{L}$.

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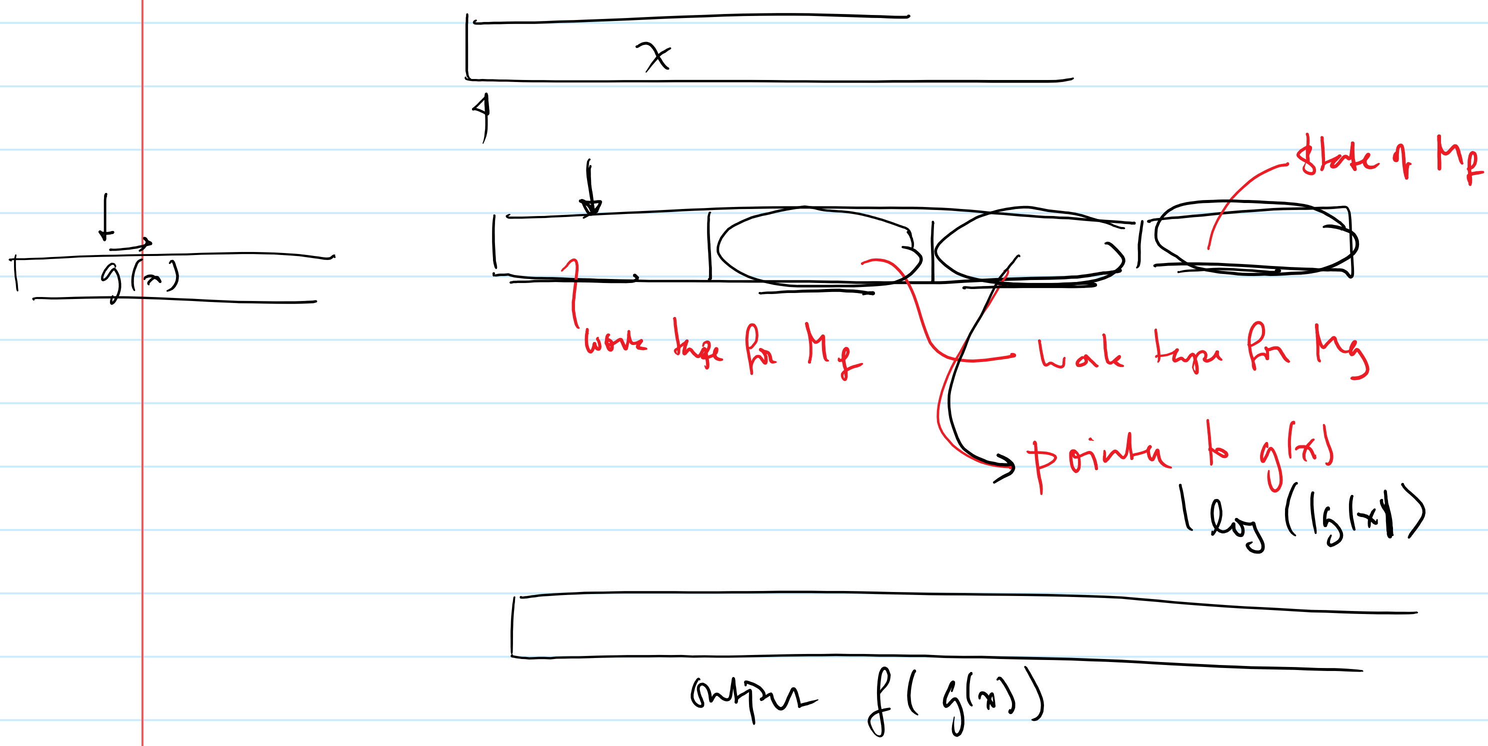

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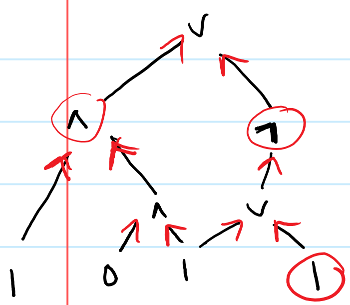
A P-complete problem.

Given a variable-free Boolean circuit.
Does it evaluate to 1?

Dfn of a circuit: Directed-Acyclic-Graph (DAG).

- exactly one sink (out-degree 0)
- Every internal node is labeled with a gate:
  - $\wedge, \vee$ (in-degree 2)
  - $\neg$ (in-degree 1)
- Every source (in-degree 0) is labeled w/ a 0 or 1.



A circuit w/ variables might have variables here.

Input: Circuit w/ no variables.
Output: Does circuit eval to 1?

Will show: $\forall$ $L \in P$ $\quad L \leq_L$ CVAL

↳ log space reductions.

Since $L \in P$, $\exists$ TM M that decides L in time n^k.

↳ will assume M halts w/ head in left-most cell.

Logspace reduction R will depend on M (but not input to x)

Input to R: x

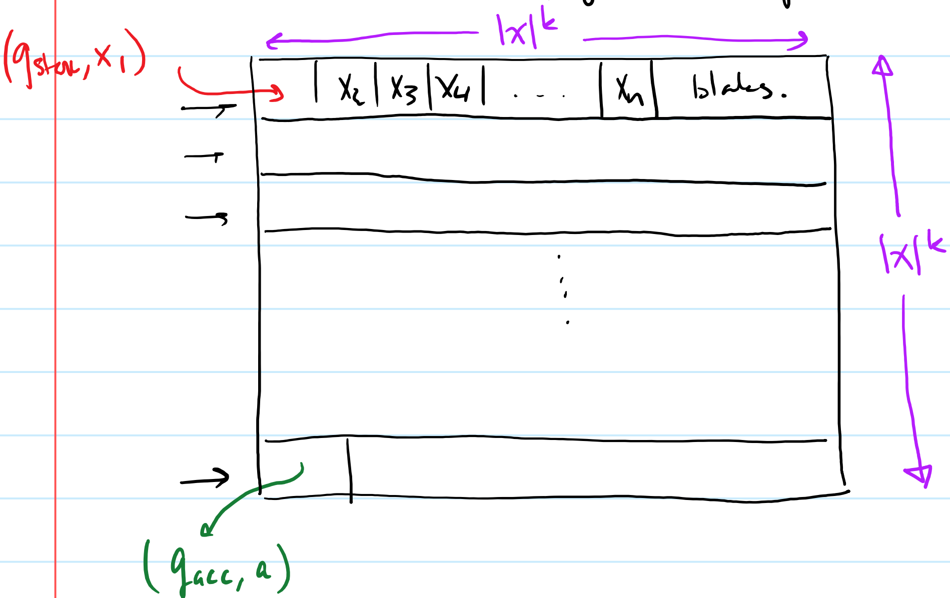
Output of R: C_x variable-free circuit.

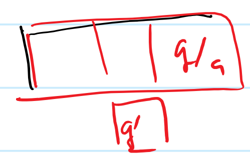
$x \in L \iff C_x$ evaluates to 1

Tableau showing computation of M on x.

Cell contents: $\sigma \in \Sigma$ or $(q, \sigma) \in Q \times \Sigma$.

Each row: Configuration of M at each time step.



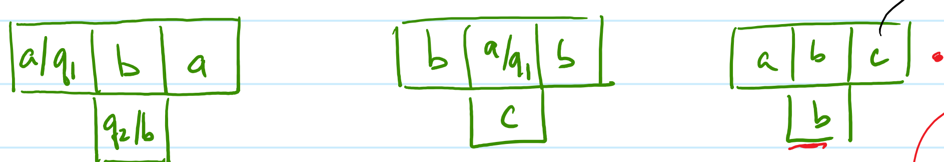


$\delta(q, a) = (\quad , L)$

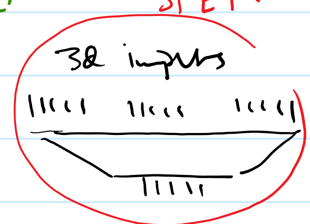
Can determine the contents of a cell given the three cells above it:

$\delta(q_1, a) \to (c, q_2, R)$.

$\lg(|\Sigma| \times |Q|) = 8$.



STEP.



Build a constant-size circuit STEP

input: binary encodings of 3 cells. → hard-coded into R's rules

output: binary encoding of cell below

STEP depends on M, but not x.

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Build poly-sized circuit using STEP as a component that simulates M's computation on x.

to get row i from row $i-1$, have $|x|^c$ copies of STEP:

row $i-1$

row i

total # copies of STEP is $|x|^{2k}$

The gates and connections which R is producing need a counter of size $O(\log |x|^{2c})$ to keep track of which copy is being output

Gate label: <row, col, gate in STEP>

Circuit Input: row 1 | q_{start}/x_1 | x_2 | x_3 | x_4 | . . . | x_n | ⊔ | ⊔ | . . . | ⊔ |

Encoding of each x_i and ⊔ are the inputs to the circuit. $\rightarrow$ $(\langle 1, i, j \rangle, x_{ij})$

row

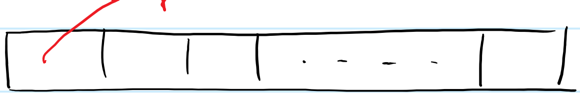
j^{th} bit of x_i.

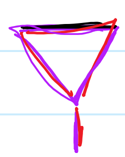
$1 \leq j \leq \log|\Sigma| + \log|Q|$.

Circuit Output:

gate?

row $|x|^k$





$\rightarrow$ circuit to determine if cell contains q_{acc}. Depends on M, not x.

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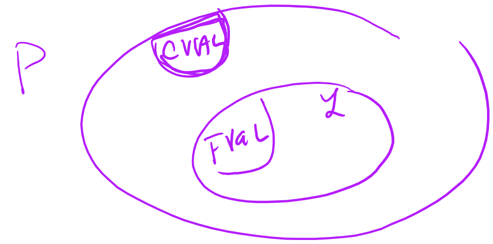

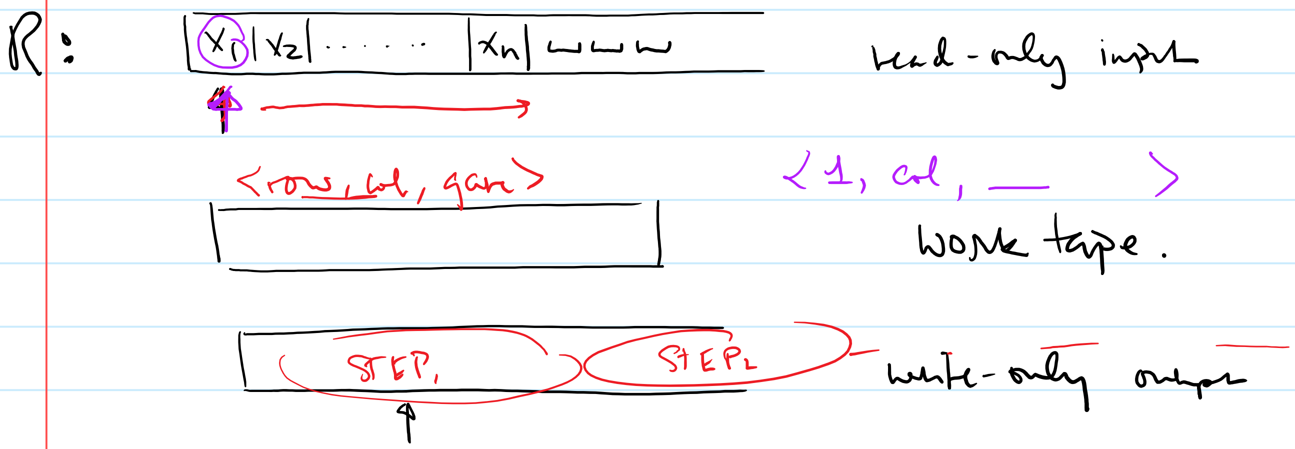

Can we evaluate CVAL in $O(\log n)$ space?
probably not.

CVAL $\in$ L iff L = P //.

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