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Sample Reduction 3SAT α Independent Set

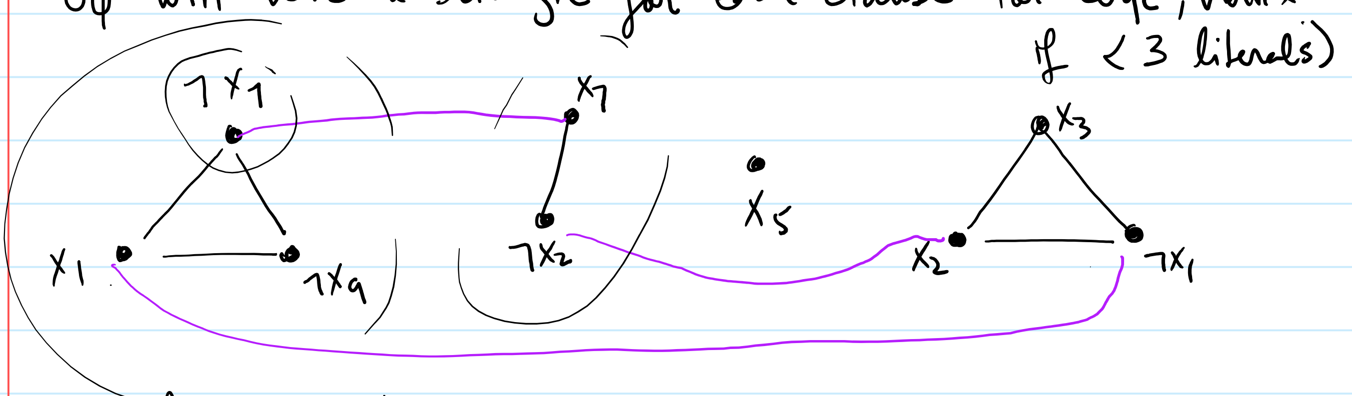
Input: Boolean formula ϕ in 3-conjunctive normal form.
Does ϕ have a satisfying assignment?

Input: undirected graph G integer k.
Does G have an indep. set of size k?
Indep Set: set of vertices with no edges between them

ex: $\underbrace{(x_1 \vee x_7 \vee \neg x_9)}_{\text{clause}} \wedge \underbrace{(\neg x_2 \vee x_7)}_{\text{clause}} \wedge (x_5) \wedge (x_2 \vee \neg x_1 \vee x_3) \ldots$

Reduction: Input : 3-CNF ϕ
Output : G_ϕ and $k = m = \#$ clauses

G_ϕ will have a triangle for each clause (or edge, vertex if < 3 literals)



Add edge between each y $\neg y$ pair.

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Indep Set of size m $\Longrightarrow$ Sat assign to ϕ

one vertex per "triangle"

Assign the corresponding literals value of T.

No contradictions (purple edges).

Sat assign for ϕ $\Longrightarrow$ Ind set of size $\geq m$

Nodes in graph that eval to true.

≥ 1 per clause.

No purple edges violated.

Pick exactly one vertex per clause.

black edges also satisfied.

$\Longrightarrow$ indep set of size m.

Completeness Complexity class C.

Language L is Complete for C

a) $L \in C$

b) for every $L' \in C$ $L' \leq L$.

L is C-hard if b) but not necessarily a).

Know that L' is C-complete and $L' \leq L$

then L is also C-complete.

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Cook in 1971 SAT is NP-complete.
Karp 1972 other problems where NP-complete.

TIME$(f(n))$

SPACE$(f(n))$

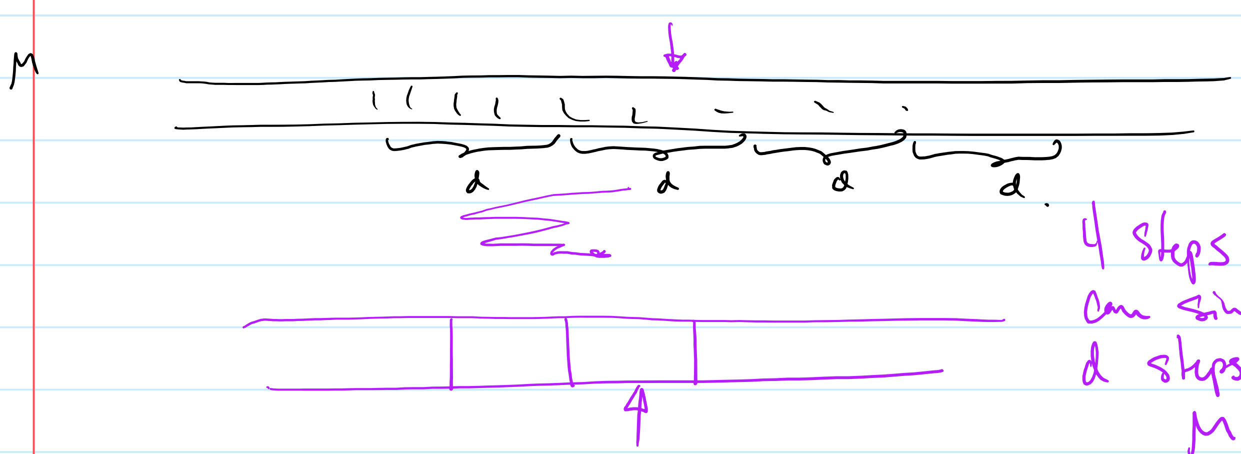
<u>Linear Speed Up</u>

If there is a TM that decides L in time $f(n)$.
Then for any $\epsilon > 0$ $\exists$ TM M' that decides L
in time $\epsilon f(n) + n + 2$.

$$\text{TIME}(f(n)) = \text{TIME}(\epsilon f(n) + n + 2).$$

$$L \in \text{TIME}(3n^2) \rightarrow L \in \text{TIME}(n^2).$$

M' has $\Sigma_{new} = \Sigma \cup \Sigma_{old}^{d}$



4 steps of M'.
can simulate
d steps of
M

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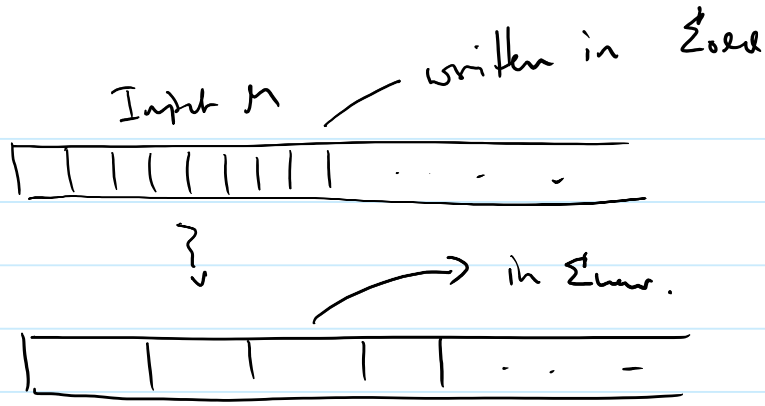
M':



translation: $n + \left\lceil \frac{n}{d} \right\rceil$ to scan back.

Running time of M: $f(n)$.

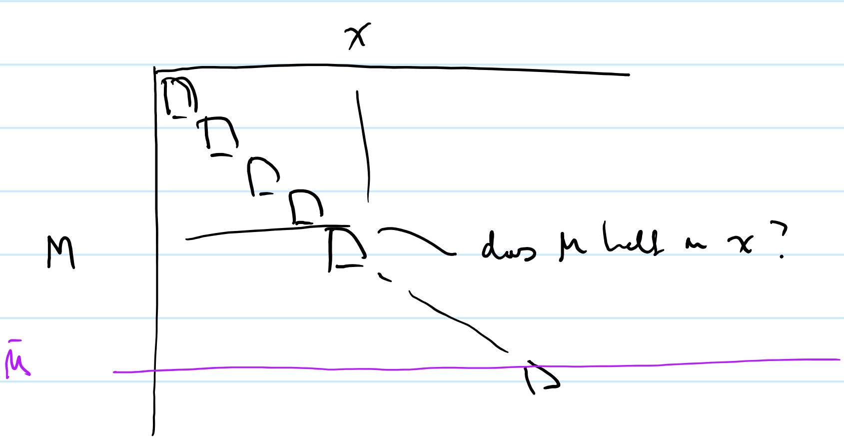
M': $n + \left\lceil \frac{n}{d} \right\rceil + 4 \cdot \frac{f(n)}{d}$.

Pick $d = \left\lceil \frac{4}{\epsilon} \right\rceil$ $\rightarrow$ $\epsilon f(n) + n + \frac{\epsilon n}{4} + 1$

When: $\mathrm{TIME}(f(n)) \subsetneq \mathrm{TIME}(g(n))$?

Halting Problem

if H decides "done"
then H'



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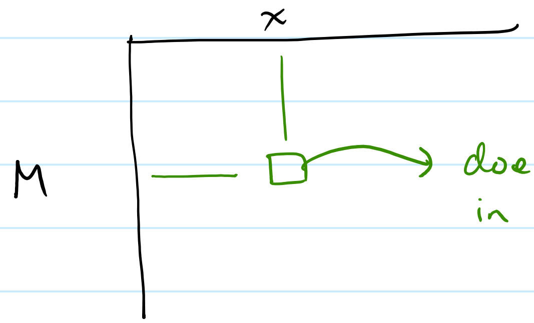

does M accept x in time $f(|x|)$?

No:

- M halts before time $f(|x|)$ & rejects
- M is still running at time $f(|x|)$.

SIM (TM) simulates M on x (input $\langle M, x\rangle$) & outputs table contents.

SIM runs in time $g(n)$.

Construct D $\quad L(D) \notin TIME(f(n))$.

D on input $\underbrace{\langle M\rangle}_{\text{length } n}$ runs SIM on $\underbrace{\langle M, M\rangle}_{\text{length } 2n}$ outputs opposite of SIM.

→ running time $g(2n)$.

Suppose $\bar{M}$ decides $L(D)$ in time $f(n)$.

$$\bar{M}(\langle \bar{M}\rangle) = SIM(\langle \bar{M}, \bar{M}\rangle) \neq D(\bar{M}).$$

$$\bar{M}(\langle \bar{M}\rangle) = D(\langle \bar{M}\rangle)$$

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Time Hierarchy: $TIME(f(n)) \subsetneq TIME((f(2n))^3)$.
$f(n) \ge n$ + is "proper" function.

Proper f: $f(n) \ge f(n-1)$

$f: \mathbb{N} \to \mathbb{N}$

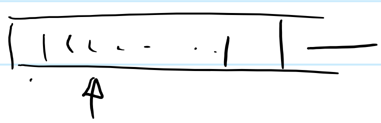
$\exists$ TM M that can output $f(n)$ symbols on input 1^n.
& runs in time $O(f(n)+n)$
& space $O(f(n))$

$\log n$ $\sqrt{n}$ n^2 2^n $n!$ proper.

$\exists$ non-proper f: $TIME(f(n)) = TIME(2^{f(n)})$

SIM: has 4 tapes on input $\langle M, x\rangle$ $|\langle M,x\rangle| = n$.

1. Contents of M's tapes.
2. M's state.
3. M's transition fcn.
4. $f(|x|)$ 1's. (clock).



Initialize: write $1^{f(|x|)}$ on tape 4.
Copy $\langle M\rangle$ to tape 3.
Keep $\langle M\rangle$ on tape 1 (input $\langle x\rangle$ to M).
Write q_{start} on tape 2.

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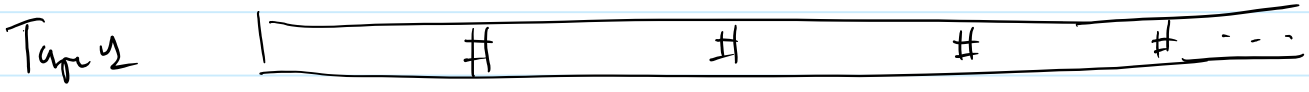
Copy current symbols under heads to tape 2.



W. $(q, \sigma_1 \sigma_2 \ldots \sigma_k)$

Scan tape 3 to implement the move.

Worst case: $O(\ell_m k_m^2 f(|x|))$

ℓ_m: length of symbol encoding

k_m: # tapes

$O(\ell_m k_m^2 f(|x|))$: 1 step of M.

$f(|x|)$ steps of M $\quad O(\underbrace{\ell_m k_m^2}_{O(n)} f^2(|x|))$.

$O(f^3(n))$.