Intro to Quantum Error Correction + Shor's 9-bit code.

Note Title 5/16/2012

We expect quantum computers to be prone to errors, so it is critical to have some means of detecting and correcting errors. This contrast with classical computers which are typically very reliable. Nonetheless there is a very rich theory of classical error corrections since errors do happen in transmitting data over communication channels.

Our discussion of quantum error correction will assume that encoding + decoding happens without error. This is OK if errors occur in transition. There is also the broader field of Fault Tolerant Quantum Computation which

* removes the assumption of perfect encoding + decoding
* shows how to implement quantum gates in a fault-tolerant way on encoded quantum states.

Classical error correction is accomplished by encoding data in a redundant way so that if part of the data gets corrupted, the problem can be detected and then corrected. This is challenging in the quantum setting because:

1) Data can not be copied (no cloning theorem)
2) Measurement destroys quantum information.
3) Errors can be continuous.

For a single qubit, one possible error is a bit flip:

$$|0\rangle \longrightarrow |1\rangle \qquad |1\rangle \rightarrow |0\rangle$$

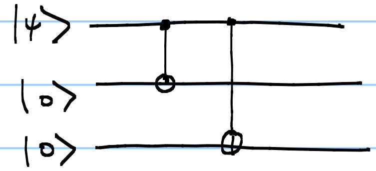
more generally: $\alpha|0\rangle + \beta|1\rangle \longrightarrow \beta|0\rangle + \alpha|1\rangle$

We will use a standard repetition code:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|0\rangle \longrightarrow |000\rangle$

$|1\rangle \longrightarrow |111\rangle$



$|\psi\rangle|00\rangle \rightarrow$

$\alpha|000\rangle + \beta|111\rangle$

We can use the following projectors to detect an error:

$$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$$
$$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$$
$$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$$
$$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$$

We can store the results in an auxiliary register:

$$\underbrace{|\;\;\;}_{\text{data}}\;\underbrace{\rangle|000\rangle}_{\text{syndrome}}$$

Can do $\quad P_1 \otimes XII + (I - P_1) \otimes I^3$

$$(\alpha|100\rangle + \beta|011\rangle)|000\rangle \longrightarrow (\alpha|100\rangle + \beta|011\rangle)|100\rangle$$

Note that measuring the syndrome does not destroy the quantum information in the 1^{st} register. It just indicates that an error has occured.

Then we can do a controlled-X : 1^{st} bit of syndrome is control
1^{st} bit of data is target.

Note that this works even if the errors happen in superposition:

Flip bit 1 w/ amplitude $1/\sqrt{2}$
Flip bit 2 w/ amplitude $1/\sqrt{2}$

$\frac{1}{\sqrt{2}}\left(\alpha|100\rangle+\beta|011\rangle\right)|100\rangle + \frac{1}{\sqrt{2}}\left(\alpha|010\rangle+\beta|101\rangle\right)|010\rangle$

Can measure syndrome before or after.

After error correction: $\left(\alpha|000\rangle+\beta|111\rangle\right)\left(\frac{1}{\sqrt{2}}|100\rangle+\frac{1}{\sqrt{2}}|010\rangle\right)$

Note this method only works if 1 or 0 qubits are flipped.

If each qubit is flipped independently with probability p:
the probability this method works is:

$$(1-p)^3 + 3p(1-p)^2 = 1-3p^2+2p^3$$

This method is an improvement over no error correction if

$$3p^2 - 2p^3 > p \quad \Rightarrow \quad p < 1/2$$

An alternative view:

Instead of performing measurement $P_0\ P_1\ P_2\ P_3$

Do two measurements: Z_1Z_2I and IZ_2Z_3

$$Z_1Z_2I = \left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)\otimes I - \left(|01\rangle\langle 01| + |10\rangle\langle 10|\right)\otimes I$$

Each observable has eigenvalues ± 1 $\Rightarrow$ total of 2 bits of information.
Z_1Z_2 compares qubits 1 + 2 (eigenvalue is +1 if the same and -1 if different).

The 2 bits of information tell ~~if a~~ where a single bit flip occurred but does not give any information about α + β.

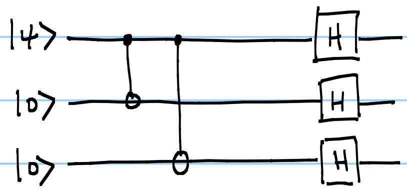
Another possible error is a phase flip:
with probability p, a Z operator is applied to $|\psi\rangle$:

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle - \beta|1\rangle.$$

Can turn a phase flip channel into a bit flip channel
Z flips $|+\rangle$ and $|-\rangle$

$$|0\rangle \longrightarrow |+++\rangle$$
$$|1\rangle \longrightarrow |---\rangle$$

Analysis is the same as with the bit flip channel. Measurements are done with respect to the $|+\rangle$ $|-\rangle$ basis.



Syndrome measurement is done by
$$H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2$$
$$H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$$

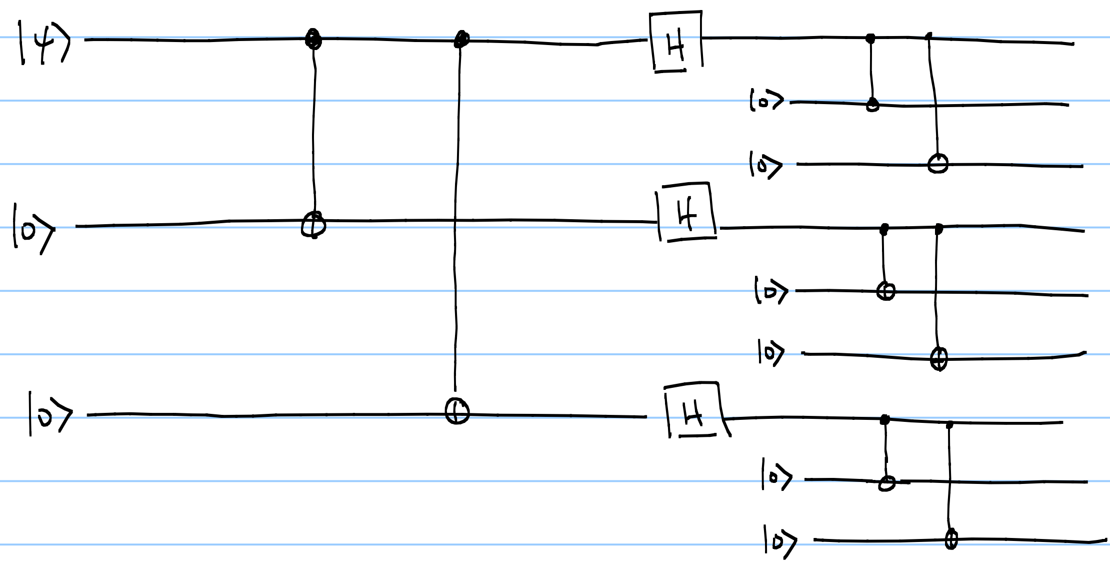
Recovery (for example) on 1^{st} qubit $H^{\otimes 3} X_1 H^{\otimes 3} = Z_1$

The Shor Code uses 9-qubits to encode a single qubit and protect against any error as long as it effects only a single qubit.

First encode the qubit using the phase-flip code then encode each of those using the bit-flip code.
This method of encoding using nested levels is called concatenation and is useful for combining the benefits of different codes.

$$|0\rangle \rightarrow \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \rightarrow \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$



To correct bit flips use the error correction procedure for bit flips in each block separately:

$Z_1Z_2 \quad Z_2Z_3 \rightarrow$ correct

$Z_4Z_5 \quad Z_5Z_6 \longrightarrow$ correct

$Z_7Z_8 \quad Z_8Z_9 \longrightarrow$ correct

Now correct for phase flips between blocks.

Note that a phase flip on 2 qubits in the same block has the same effect.

Measure $\quad X_1X_2X_3X_4X_5X_6$

$\quad\quad\quad\quad X_4X_5X_6X_7X_8X_9$

To correct a single phase flip in the first block, apply $Z_1Z_2Z_3$

$Z_1 Z_2 Z_3$: $\alpha\,(|000\rangle - |111\rangle) + \beta\,(|000\rangle + |111\rangle)$

$$\rightarrow \alpha\,(|000\rangle + |111\rangle) + \beta\,(|000\rangle - |111\rangle)$$

If error is both a bit flip and a phase flip: $X_1 Z_1$
each error correction procedure (bit flip then phase flip)
will work independently.

In fact any 1-qubit error can be expressed as:

$$E = e_0 I + e_1 X_1 + e_2 Z_1 + e_3 X_1 Z_1$$

$$E|\psi\rangle = e_0|\psi\rangle + e_1 X_1|\psi\rangle + e_2 Z_1|\psi\rangle + e_3 X_1 Z_1|\psi\rangle$$

Measuring the syndrome collapses to one of the four states:

$$|\psi\rangle \quad X_1|\psi\rangle \quad Z_1|\psi\rangle \quad X_1 Z_1|\psi\rangle$$

and the appropriate recovery can be applied to get back to $|\psi\rangle$.

A whole continuum of errors can be handled by
correcting only a discrete set!