Phase Estimation.

Note Title 5/1/2012

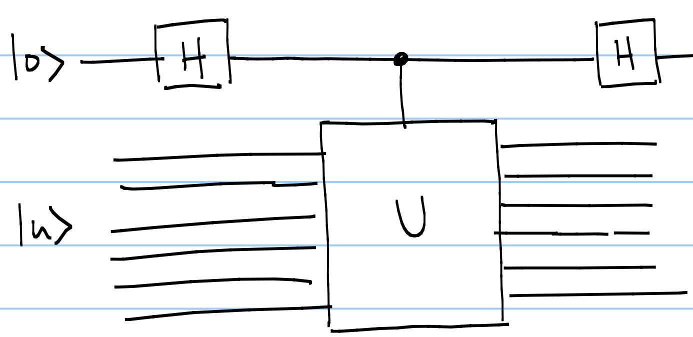
Another algorithm that makes essential use of the QFT is phase estimation.

Suppose we have a unitary operator U with eigenvector $|u\rangle$ and eigenvalue $e^{2\pi i \varphi}$ where φ is unknown. "λ"
The goal is to estimate φ. (actually will just get $|\varphi|$).
We are given $|u\rangle$ as input and a set of black boxes that can compute controlled - U^{2^j} for some set of non-negative integers j.

Phase estimation is an ingredient in numerous other quantum algorithms - including approximate counting. Whether we actually have these black-boxes will depend on the application in which it is being used.

Lets start with C-U a controlled U operation.

Consider the following circuit on input $|0\rangle|u\rangle$



$$|0\rangle|u\rangle \xrightarrow{H} |+\rangle|u\rangle$$

$$\xrightarrow{C\text{-}U} \frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle + |1\rangle\lambda|u\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + \lambda|1\rangle\right)|u\rangle$$

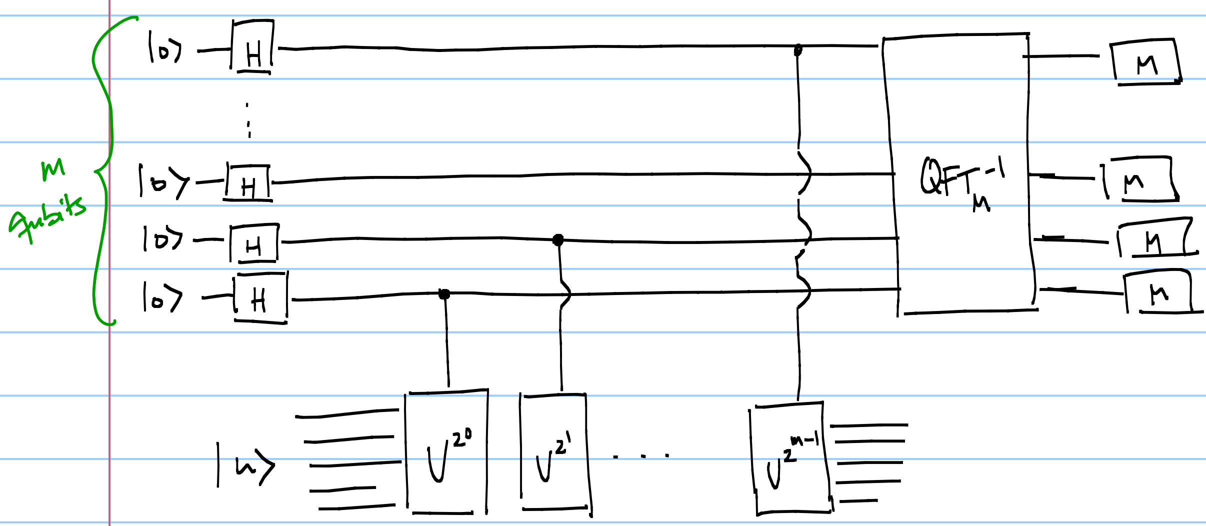
$$\xrightarrow{H} \left[\frac{1+\lambda}{\sqrt{2}}|0\rangle + \frac{1-\lambda}{\sqrt{2}}|1\rangle\right]|u\rangle$$

After measurement: $\quad \text{prob of } 0 = \left|\frac{1+\lambda}{\sqrt{2}}\right|^2 = \frac{1+\cos(2\pi\varphi)}{2}$

$\quad \text{prob of } 1 = \left|\frac{1-\lambda}{\sqrt{2}}\right|^2 = \frac{1-\cos(2\pi\varphi)}{2}$

Bias of the coin is $\cos(2\pi\varphi)/2$
However to estimate $\cos(2\pi\varphi)/2$ to within m bits of accuracy, we need to repeat the experiment $2^{\Omega(m)}$ times. Estimating the bias of a coin to within ϵ with probability δ requires $O(\log(1/\delta)/\epsilon)$ Samples. The following generalization still has complexity 2^m but doesn't require the repeated measurements.

Estimating φ to within m bits of accuracy is the same as finding the integer j such that $j/2^m$ is the best approximation of φ. Let $M = 2^m$ and $\omega_M = e^{(2\pi i/M)}$



$$|0^m\rangle|u\rangle \xrightarrow{H^{\otimes m}} \left(\frac{1}{\sqrt{M}}\sum_{k=0}^{M-1}|k\rangle\right)\otimes|u\rangle$$

$$\xrightarrow{C_m - U} \left(\frac{1}{\sqrt{M}} \sum_{k=0}^{n-1} \lambda^k |k\rangle \right) \otimes |u\rangle$$

Now suppose $\lambda = e^{2\pi i\, j/M}$

(Actually $\lambda = e^{2\pi i \varphi}$ and j/M is the best approximation of θ to within $\pm \frac{1}{M}$)

$$= \underbrace{\left[\frac{1}{\sqrt{M}} \sum_{k=0}^{n-1} \left[e^{\frac{2\pi i}{M}} \right]^{jk} |k\rangle \right]}_{} \otimes |u\rangle$$

Note that this is $FT_M |j\rangle$

So if we apply $(FT_M)^{-1}$ we get...

$$\xrightarrow{(FT_M)^{-1}} |j\rangle |u\rangle$$

If we only have $\varphi \approx j/M$, the output is j with high probability.

The exact statement for the error which we state here without proof is that to successfully obtain φ accurate to t bits with probability $1 - \epsilon$ requires

$$m = t + \left\lceil \log\left(2 + \frac{1}{2\epsilon}\right) \right\rceil$$