Tensor Products

Note Title 4/8/2012

Suppose that we have two quantum systems with Hilbert spaces: $V : \mathbb{C}^k$ (k distinguishable states)
$W : \mathbb{C}^l$ (l distinguishable states)

What is the Hilbert space for the composite system? (Note that the composite system has $k \cdot l$ distinguishable states.)

Hilbert Space $V \otimes W$ ("V tensor W")

Suppose we have: Basis for V: $|v_1\rangle, |v_2\rangle, \ldots, |v_k\rangle$
Basis for W: $|w_1\rangle, |w_2\rangle, \ldots, |w_l\rangle$

A basis for $V \otimes W$ will be: $|v_i\rangle \otimes |v_j\rangle$ $\quad 1 \le i \le k$, $1 \le j \le l$

An arbitrary state $|\phi\rangle$ in $V \otimes W$ can be expressed as:

$$|\phi\rangle = \sum_{\substack{1 \le i \le k \\ 1 \le j \le l}} \alpha_{ij} |v_i\rangle \otimes |v_j\rangle$$

Inner product extends to states in $V \otimes W$ as:

$$(|v_i\rangle \otimes |w_i\rangle, |v_j\rangle \otimes |v_j\rangle) = \langle v_i | v_j \rangle \langle w_i | w_j \rangle$$

This is extendable to arbitrary states by linearity.

We have already seen examples of tensor products when we looked a systems of n qubits.

Hilbert space for one qubit: $\mathbb{C}^2$
Hilbert space for n-qubits: $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$

Note that when we add a qubit, we double the number of complex numbers required to specify an arbitrary state.

We can define linear operators on a composite system by taking the tensor product of two operators, each acting on a subsystem ... an linear operators on those.

A_V acts on V

A_W acts on W

$$(A_V \otimes A_W)\, |v\rangle \otimes |w\rangle = A_V |v\rangle \otimes A_W |w\rangle$$

The action of $A_V \otimes A_W$ on an arbitrary state in $V \otimes W$ is defined by linearity.

For example: $V = \mathbb{C}^2$ (one qubit) basis $\{|0\rangle, |1\rangle\}$
$W = \mathbb{C}^2$ (another qubit) basis $\{|0\rangle, |1\rangle\}$

$V \otimes W = \mathbb{C}^2 \otimes \mathbb{C}^2$ Basis for $V \otimes W$: $|0\rangle \otimes |0\rangle = |00\rangle$

$|0\rangle \otimes |1\rangle = |01\rangle$

$|1\rangle \otimes |0\rangle = |10\rangle$

$|1\rangle \otimes |1\rangle = |11\rangle$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad |0\rangle\langle 0| - |1\rangle\langle 1|$$

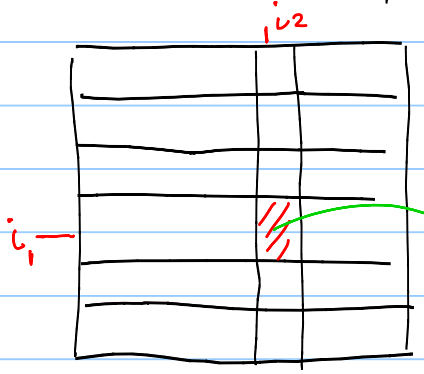
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$(Z \otimes X)\, |10\rangle = \underbrace{Z|1\rangle}_{-|1\rangle} \otimes \underbrace{X|0\rangle}_{|1\rangle} = -|11\rangle$$

What does the matrix representation of $A_V \otimes A_W$ look like? Pick a basis for V + W. Express A_V + A_W in matrix form $(\underbrace{M_V}_{k\times k} + \underbrace{M_W}_{l\times l})$

$M_V \otimes M_W$ is a $kl \times kl$ matrix

$$(M_V \otimes M_W)_{i_1 k + j_1,\, i_2 k + j_2} = [M_V]_{i_1 i_2} \cdot [M_W]_{j_1 j_2}.$$



$k \times k$ blocks, each of size $l \times l$.

$$\underbrace{[M_V]_{i_1 i_2}}_{\text{scalar}} \cdot \underbrace{[M_W]}_{l \times l \text{ matrix}}$$

In our example above

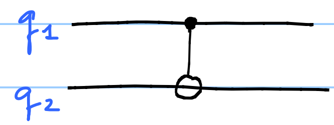
$$Z \otimes X = \begin{bmatrix} 1\cdot[X] & 0[X] \\ 0[X] & -1[X] \end{bmatrix} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 1 & 0 & 0 \\ 01 & 1 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & -1 \\ 11 & 0 & 0 & -1 & 0 \end{array}$$

We will use this a lot when we talk about gates and circuits. If we have n-qubits on which we are performing a computation, we typically only operate on 2 qubits at a time. Thus, a gate is specified by a 4×4 matrix but it is really operating on a big Hilbert space of dimension 2^n, so any operator must be a $2^n \times 2^n$ matrix. The gate matrix is really tensored with the identity on the rest of the qubits.

A 2-qubit gate we will encounter a lot is a CNOT gate

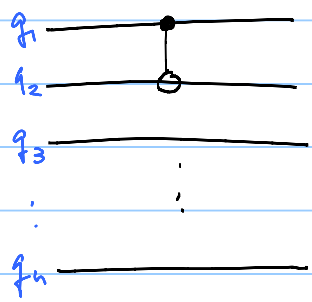
$$\begin{array}{cc} & q_1 q_2 \\ q_1 q_2 & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

denoted by:



q_1 is the control bit if $q_1 = 0$, q_2 is unchanged
q_2 is the target bit if $q_1 = 1$, q_2 is flipped

if we have



this is really:

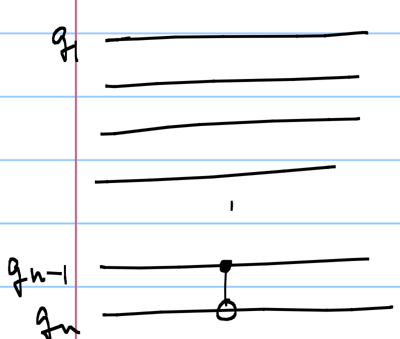
$$[CNOT]_{12} \otimes I_{3 \dots n}$$

$$\begin{matrix} [0] & [I] & [0] & [0] \\ [I] & [0] & [0] & [0] \\ [0] & [0] & [0] & [I] \\ [0] & [0] & [I] & [0] \end{matrix}$$

all 0's $2^{n-2} \times 2^{n-2}$

$2^{n-2} \times 2^{n-2}$ identity

If instead we had $I_{1 \dots n-2} \otimes [CNOT]_{n-1,n}$ we would have:



$$\begin{bmatrix} \boxed{C} & & & \\ & \boxed{C} & & 0 \\ & & \ddots_{2^{n-2}} & \\ 0 & & & \boxed{C} \end{bmatrix}$$

4x4 CNOT

Here are some examples of 1-qubit gates that we will frequently encounter:

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

this flips between 0/1 + +/- bases:

$H|0\rangle = |+\rangle \qquad H|+\rangle = |0\rangle$

$H|1\rangle = |-\rangle \qquad H|-\rangle = |1\rangle$

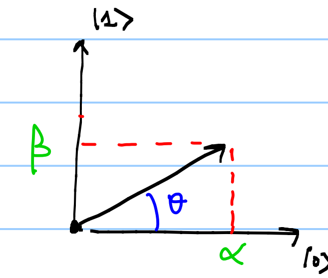
$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ phase flip

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$ not

Rotate by θ

$$U_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(for real-valued amplitudes)



$\alpha|0\rangle + \beta|1\rangle$

$\hookrightarrow U_\theta(\cos\varphi|0\rangle + \sin\varphi|1\rangle) = \cos(\theta+\varphi)|0\rangle + \sin(\theta+\varphi)|1\rangle$

Tensor Product for States

$$|\psi_v\rangle = \sum_{i=1}^{k} \alpha_i |v_i\rangle \qquad |\psi_w\rangle = \sum_{j=1}^{\ell} \beta_j |w_j\rangle$$

$$|\psi_v\rangle \otimes |\psi_w\rangle = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \alpha_i \beta_j |v_i\rangle \otimes |w_j\rangle$$

For example, suppose we have 2 qubits

$$|\psi_1\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$
$$|\psi_2\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Not every state on 2 qubits can be expressed as a tensor product:

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$ entangled state

This has important implications for measurement

In the case $|\psi_1\rangle \otimes |\psi_2\rangle$ if we measure first qubit

if the outcome is 0 state afterwards is $|0\rangle |\psi_2\rangle$

if the outcome is 1 state afterwards is $|1\rangle |\psi_2\rangle$

Measuring qubit 1 gives us no information about qubit 2.

Now what if we measure the first qubit of the entangled state $1/\sqrt{2}\,(|00\rangle + |11\rangle)$.

outcome is 0 w.p. $1/2$ $\longrightarrow$ $|00\rangle$

outcome is 1 w.p. $1/2$ $\longrightarrow$ $|11\rangle$

After measuring the first qubit we know the value of the second qubit with certainty (measured in standard basis). – even if qubits are physically separated