Tensor Products - 1

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Tensor Products:

Suppose that we have two quantum systems with Hilbert spaces

$V : \mathbb{C}^k$ (dimension k)

$W : \mathbb{C}^\ell$ (dimension ℓ)

What is the Hilbert space of the composite system?

Composite system will have dimension $k \cdot \ell$

Hilbert space $V \otimes W$ ("V tensor W")

Suppose Basis for V: $|v_1\rangle \; |v_2\rangle \; \dots \; |v_k\rangle$

Basis for W: $|w_1\rangle \; |w_2\rangle \; \dots \; |w_\ell\rangle$

A basis for $V \otimes W$ will be: $|v_i\rangle \otimes |w_j\rangle$ $\quad 1 \le i \le k$, $1 \le j \le \ell$.

An arbitrary state in $V \otimes W$ can be expressed as:

$$|\phi\rangle = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \alpha_{ij} \, |v_i\rangle \otimes |w_j\rangle$$

$k \cdot \ell$ complex numbers used to describe a generic state in $V \otimes W$.

Tensor Products - 2

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Inner Product of $|v_i\rangle \otimes |w_j\rangle$ and $|v_a\rangle \otimes |w_b\rangle$ is:

$$\langle v_i | v_a\rangle \langle w_j | w_b\rangle$$

$=1$ if $i=a$
$=0$ if $i\neq a$.

If $\{|v_i\rangle\}$ is orthonormal $\langle v_i | v_a\rangle = \delta_{ia}$
If $\{|w_i\rangle\}$ is orthonormal $\langle w_j | w_b\rangle = \delta_{jb}$

$$\Rightarrow \quad \langle v_i | v_a\rangle \langle w_j | w_b\rangle = \delta_{ia}\cdot\delta_{jb}$$

(0 unless both indices match)

$|v_1\rangle|w_1\rangle$ is orthogonal to $|v_1\rangle|w_2\rangle$ and $|v_2\rangle|w_1\rangle$

$\Rightarrow \{|v_i\rangle \otimes |w_j\rangle\}_{\substack{1\le i\le l \\ 1\le j\le k}}$ is an orthonormal basis.

Inner product of arbitrary states extendable by linearity:

$$|\phi\rangle = \frac{1}{\sqrt{2}}|v_1\rangle \otimes |w_1\rangle + \frac{i}{\sqrt{2}}|v_2\rangle \otimes |w_2\rangle$$

$$|\psi\rangle = i/\sqrt{3}\,|v_1\rangle \otimes |w_1\rangle + \sqrt{\tfrac{2}{3}}\,|v_1\rangle \otimes |w_2\rangle$$

$\langle\psi|\phi\rangle = -i/\sqrt{3}\cdot 1/\sqrt{2}\,\langle v_1|v_1\rangle\langle w_1|w_1\rangle$
$\quad -i/\sqrt{3}\cdot i/\sqrt{2}\,\langle v_1|v_2\rangle\langle w_1|w_2\rangle$
$\quad \sqrt{2/3}\cdot 1/\sqrt{2}\,\langle v_1|v_1\rangle\langle w_2|w_1\rangle$
$\quad \sqrt{2/3}\cdot i/\sqrt{2}\,\langle v_1|v_2\rangle\langle w_2|w_2\rangle = -i/\sqrt{6}.$

$\langle\psi$

Tensor Products - 3

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n qubits $\mathbb{C}^{2^n}$

$+ 1$ qubit $\mathbb{C}^2$

$\mathbb{C}^{2^n} \otimes \mathbb{C}^2$

$\hookrightarrow$ dim $2^n \cdot 2 = 2^{n+1}$

We have already seen tensor products:

Hilbert space for 1 qubit: $\mathbb{C}^2$

Hilbert space for n qubits: $\underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \mathbb{C}^2}_{n \text{ times}} \cong \mathbb{C}^{2^n}$

Note: When we add 1 qubit, we double the length of the vectors to describe a state.

Example: $V = \mathbb{C}^2$ (left qubit) basis $\{|0\rangle, |1\rangle\}$

$W = \mathbb{C}^2$ (right qubit) basis $\{|0\rangle, |1\rangle\}$

$V \otimes W = \mathbb{C}^2 \otimes \mathbb{C}^2$ Basis for $V \otimes W$: $|0\rangle \otimes |0\rangle = |00\rangle$

$|0\rangle \otimes |1\rangle = |01\rangle$

$|1\rangle \otimes |0\rangle = |10\rangle$

$|1\rangle \otimes |1\rangle = |11\rangle$

Tensor Products of States:

$$|\psi_v\rangle = \sum_{i=1}^{k} \alpha_i |v_i\rangle \qquad |\psi_w\rangle = \sum_{j=1}^{l} \beta_j |w_j\rangle$$

$$|\psi_v\rangle \otimes |\psi_w\rangle = \sum_{i=1}^{k} \sum_{j=1}^{l} \alpha_i \cdot \beta_j \; |v_i\rangle \otimes |w_j\rangle$$

specified by $k + l$ complex numbers

(a generic state in $V \otimes W$ requires $k \cdot l$ complex numbers)

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For example suppose we have 2 qubits:

$$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$
$$|\psi_b\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

$$|\psi_a\rangle \otimes |\psi_b\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Viewed as vectors:

$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \otimes \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} = \begin{bmatrix}\alpha_0\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}\\ \alpha_1\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{bmatrix}$$

More Generally:

$$\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_k\end{bmatrix} \otimes \begin{bmatrix}\beta_1\\ \vdots\\ \beta_\ell\end{bmatrix} = \begin{bmatrix}\alpha_1\begin{bmatrix}\beta_1\\ \vdots\\ \beta_\ell\end{bmatrix}\\ \alpha_2\begin{bmatrix}\beta_1\\ \vdots\\ \beta_\ell\end{bmatrix}\\ \vdots\\ \alpha_k\begin{bmatrix}\beta_1\\ \vdots\\ \beta_\ell\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\alpha_1\beta_1\\ \vdots\\ \alpha_k\beta_\ell\end{bmatrix}$$

$$|\psi_a\rangle \otimes |\psi_b\rangle = \sum_{\substack{1\le i\le k\\ 1\le j\le \ell}} \alpha_i\beta_j\, |v_i\rangle \otimes |w_j\rangle$$

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An alternate basis for a 2-qubit system is:

$$|+\rangle \otimes |+\rangle$$
$$|+\rangle \otimes |-\rangle$$
$$|-\rangle \otimes |+\rangle$$
$$|-\rangle \otimes |-\rangle$$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

$$|+\rangle \otimes |-\rangle = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \\ \\ 1/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

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Not every 2-qubit state can be expressed as:

$$|\psi_a\rangle \otimes |\psi_b\rangle$$

For example: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is an entangled state.

This has important implications for measurement.

Consider again:

$$|\psi_a\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \qquad |\alpha_0|^2 + |\alpha_1|^2 = 1$$
$$|\psi_b\rangle = \beta_0|0\rangle + \beta_1|1\rangle \qquad |\beta_0|^2 + |\beta_1|^2 = 1.$$

$$|\psi_a\rangle \otimes |\psi_b\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Now measure the first qubit:

Probability of 0: $|\alpha_0\beta_0|^2 + |\alpha_0\beta_1|^2$

$$= |\alpha_0|^2|\beta_0|^2 + |\alpha_0|^2|\beta_1|^2$$
$$= |\alpha_0|^2(|\beta_0|^2 + |\beta_1|^2) = |\alpha_0|^2$$

Afterwards state is: $\dfrac{\alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle}{[\underbrace{|\alpha_0\beta_0|^2 + |\alpha_0\beta_1|^2}_{|\alpha_0|^2}]^{1/2}}$

$$\frac{\alpha_0}{|\alpha_0|}(\beta_0|00\rangle + \beta_1|01\rangle)$$
$$= \frac{\alpha_0}{|\alpha_0|}|0\rangle \otimes (\beta_0|0\rangle + \beta_1|1\rangle) = |0\rangle \otimes |\psi_b\rangle$$

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If a 2-qubit system is in a tensor product state $|\psi_a\rangle \otimes |\psi_b\rangle$ and the first qubit is measured then:

- prob of different outcomes depends only on $|\psi_a\rangle$
- afterwards, state is $|0\rangle \otimes |\psi_b\rangle$ or $|1\rangle \otimes |\psi_b\rangle$.

(state of second qubit unchanged).
(no info gained about the state of the 2^{nd} qubit).

Now consider $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (not a tensor product state)

measure 1^{st} qubit: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

outcome $= 0$ w.p. $1/2$ → $|00\rangle$

outcome $= 1$ w.p. $1/2$ → $|11\rangle$

After measuring the first qubit we know the value of the 2^{nd} qubit with certainty.
(even if qubits are physically separated).

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Linear Operators on Composite Systems

A_V acts on V

A_W acts on W

$$(A_V \otimes A_W)\,|v\rangle \otimes |w\rangle = A_V|v\rangle \otimes A_W|w\rangle$$

Example: 2-qubit system

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \left(|0\rangle\langle 0| - |1\rangle\langle 1| \right)|1\rangle$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad |1\rangle\langle 0| + |0\rangle\langle 1|$$

$$Z \otimes X\,|1\,0\rangle = Z|1\rangle \otimes X|0\rangle$$

$$-|1\rangle \otimes |1\rangle = -|11\rangle$$

What does the matrix representation look like?

$$Z \otimes X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$Z \otimes X \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$|10\rangle \qquad -|11\rangle$

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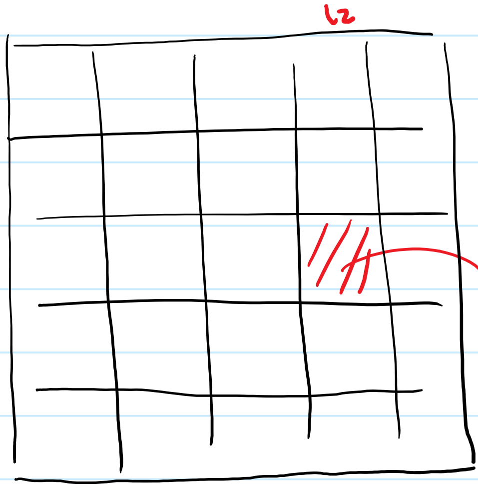
More generally $\overbrace{A_v}^{k \times k \text{ matrix}} \otimes \overbrace{A_w}^{l \times l \text{ matrix}}$ is a linear operator on $\mathbb{C}^{k \cdot l}$ $\Rightarrow$ $k \cdot l \times k \cdot l$ matrix

$$(A_v \otimes A_w)_{i_1 \cdot k + j_1,\ i_2 \cdot k + j_2} = [A_v]_{i_1, i_2} \cdot [A_w]_{j_1, j_2}$$

i_2

i_1



$k \times k$ blocks each of size $l \times l$

$$\underbrace{[A_v]_{i_1, i_2}}_{\text{scalar}} \cdot \underbrace{[A_w]}_{l \times l \text{ matrix}}$$

$$Z \otimes X = \begin{bmatrix} 1\,[X] & 0 \cdot [X] \\ 0 \cdot [X] & -1\,[X] \end{bmatrix} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 1 & 0 & 0 \\ 01 & 1 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & -1 \\ 11 & 0 & 0 & -1 & 0 \end{array}$$

Tensor Products - 10

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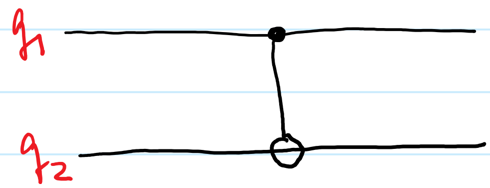
We will talk about "gates" that act on a pair of qubits in a larger n-bit system

Example: CNOT gate.



q_1 is the control bit. if $q_1 = 0$ q_2 doesn't change

if $q_1 = 1$ q_2 is flipped

$$
\begin{array}{cc}
 & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\
\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
\end{array}
$$

q_1 q_2

$\text{CNOT}\,|00\rangle \quad |00\rangle$

$|01\rangle \quad |01\rangle$

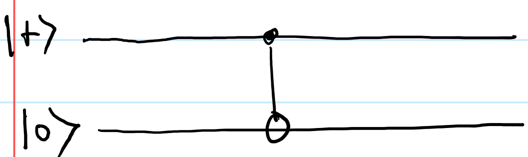
$|10\rangle \quad |11\rangle$

$|11\rangle \quad |10\rangle$

What does CNOT do to the state:

$$|\phi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

$$\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_3|10\rangle + \alpha_2|11\rangle$$



$\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$

$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$|+\rangle|0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

$$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

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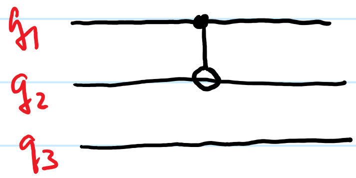
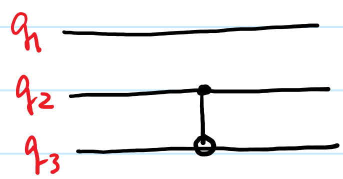


This is really:

$$[CNOT]_{12} \otimes I_3$$

$$\underbrace{\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}}_{[CNOT]_{12}} \otimes \underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{I_3} = \begin{bmatrix} \begin{matrix}[I_3] & [0]\\ [0] & [I_3]\end{matrix} & [\,0\,] \\ [\,0\,] & \begin{matrix}[0] & [I_3] \\ [I_3] & [0]\end{matrix}\end{bmatrix}$$

What about:



$$[I]_1 \otimes [CNOT]_{23} = \begin{bmatrix} 1\cdot[CNOT] & 0\cdot[\,0\,] \\ 0\cdot[\,0\,] & 1\cdot[CNOT] \end{bmatrix}$$

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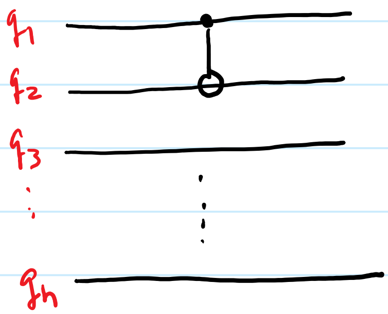
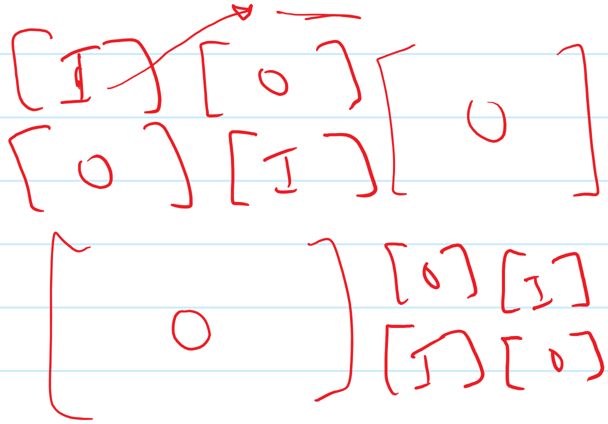
More generally:

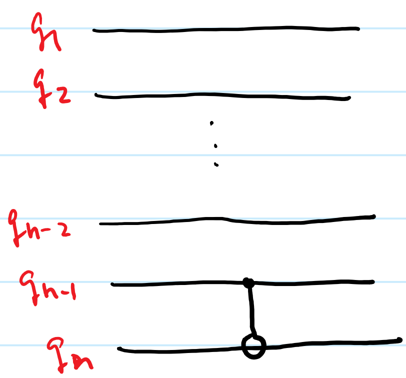


dimension? $2^{n-2} \otimes 2^{n-2}$

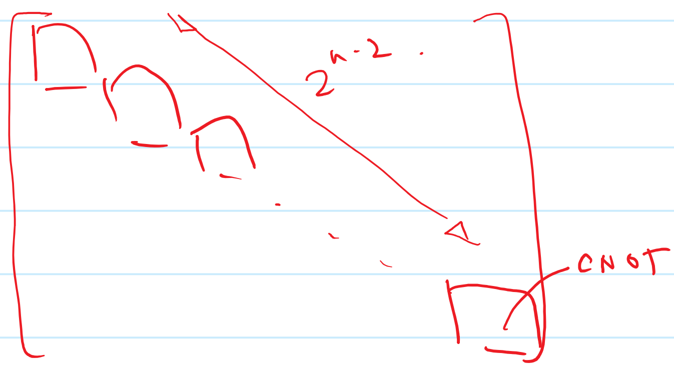
4×4

$[CNOT]_{12} \otimes I_{3\dots n}$





$I_{1\dots n-2} \; [CNOT]_{n-1,n}$



Tensor Products - 13

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Here are some common 1-qubit gates:

Hadamard $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ Flips between 0/1 and +/- bases

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$H|0\rangle = |+\rangle$ $H|+\rangle = |0\rangle$ $H \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$H|1\rangle = |-\rangle$ $H|-\rangle = |1\rangle$

$H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $H\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ phase flip. $Z|+\rangle = |-\rangle$

$Z|-\rangle = |+\rangle$

$Z|0\rangle = |0\rangle$ $Z|1\rangle = -|1\rangle$

$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ NOT

$X|0\rangle = |1\rangle$ $X|1\rangle = |0\rangle$.