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No cloning theorem:

Making copies of data is straight-forward on a classical computer.

Quantum Information is more subtle.

Suppose we want a procedure that can copy a generic qubit.

What does that look like?

$$|\phi\rangle|s\rangle \longrightarrow |\phi\rangle|\phi\rangle$$

- $|\phi\rangle$ is the input state that we want to copy.

  Procedure should work for any $|\phi\rangle$
- $|s\rangle$ is a generic state that will be over-written.

  $|s\rangle$ will be the same for every input.
- Operation must be unitary.

$\alpha|0\rangle + \beta|1\rangle$

$|0\rangle \quad |1\rangle$

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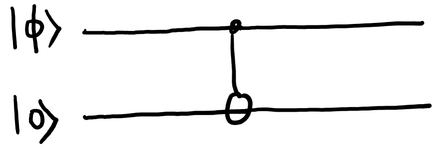
$|\phi\rangle|0\rangle = \alpha|00\rangle + \beta|10\rangle$

$\downarrow$ CNOT

$\alpha|00\rangle + \beta|11\rangle$

Consider this operation:



if $|\phi\rangle = \underline{\alpha}|0\rangle + \underline{\beta}|1\rangle$ the output is:

$\alpha|00\rangle + \beta|11\rangle$

$|\phi\rangle|\phi\rangle$

$\Big\}\, M \ne I$

$|0\rangle|\phi\rangle$

Which is not the same as:

$$\Rightarrow |\phi\rangle|\phi\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

No Cloning Theorem: there no qubit state $|s\rangle$ and unitary operation U such that

$$\underline{U}(|\phi\rangle\underline{|s\rangle}) = \underline{|\phi\rangle|\phi\rangle}$$

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for every qubit state $|\phi\rangle$.

Proof Consider $|\psi\rangle$ $|\phi\rangle$ such that $\langle\psi|\phi\rangle \ne 0$

$\langle\psi|\phi\rangle \ne 1$.

$$\rightarrow U(|\phi\rangle|s\rangle) = \underline{|\phi\rangle|\phi\rangle}$$

$$\rightarrow U(\underline{|\psi\rangle}|s\rangle) = \underline{|\psi\rangle|\psi\rangle}.$$

$$\langle\phi| \otimes \langle s|\, U^{\dagger} U\, |\psi\rangle \otimes |s\rangle = \langle\phi| \otimes \langle\phi|\; |\psi\rangle \otimes |\psi\rangle$$

$$\langle \phi | \psi \rangle \cdot \langle s | s \rangle = \langle \phi | \psi \rangle \langle \phi | \psi \rangle$$

$$\langle \phi | \psi \rangle = \langle \phi | \psi \rangle^2 .$$

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Quantum Teleportation.

The no cloning theorem tells us that there is no unitary transformation that does

$$|\psi\rangle \otimes |0\rangle \rightarrow |\psi\rangle \otimes |\psi\rangle$$

for all $|\psi\rangle$.

However if we are willing to destroy the original then it is possible to transmit a qubit using an entangled pair + 2 bits of classical communication.

Alice would like to transmit $|\psi\rangle$ (single qubit state) to Bob. Alice and Bob share the entangled state

$$|\Phi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

(Alice has first qubit and Bob has the second.)

$$\underbrace{|\psi\rangle \otimes |\Phi}_{\text{Alice}}\underbrace{\rangle}_{\text{Bob}} \rightsquigarrow |0\rangle|0\rangle\underbrace{|\psi\rangle}_{\text{Bob.}}$$

Alice $|\psi\rangle$
1st qubit of $|\Phi\rangle$

Bob 2nd qubit of $|\Phi\rangle$

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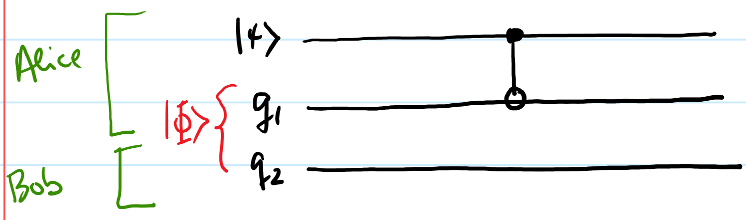
$$|\psi\rangle = \underline{a_0|0\rangle} + a_1|1\rangle$$



Another view of the CNOT gate.

$$\text{CNOT}(xy) = x\,(x \oplus y)$$

$00 \to 0\,(0 \oplus 0) = 00$

$01 \to \underline{0}\,(0 \oplus 1) = 01$

$10 \to 1\,(1 \oplus 0) = 11$

$11 \to 1\,(1 \oplus 1) = 10.$

Start with $|\psi\rangle \otimes |\Phi\rangle = (a_0|0\rangle + a_1|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$= \sum_{\substack{i=0,1 \\ j=0,1}} \frac{a_i}{\sqrt{2}} |i\rangle |jj\rangle$$

$$|i\rangle |j\rangle \underline{|j\rangle}$$

After CNOT, this becomes: $\displaystyle\sum_{\substack{i=0,1 \\ j=0,1}} \frac{a_i}{\sqrt{2}} |\underline{i}\rangle |\underline{i \oplus j}\rangle |j\rangle$

Then Alice measures the middle qubit.

Outcome $= 0$: $\quad i \oplus j = 0 \Rightarrow (i = j)$ $\quad \displaystyle\sum_{j=0,1} \frac{a_j}{\sqrt{2}} |j\rangle |0\rangle |j\rangle$

renormalize.

$$a_0|000\rangle + a_1|101\rangle$$

Outcome $= 1$: $\quad i \oplus j = 1 \quad (i \neq j)$ $\quad a_0|011\rangle + a_1|110\rangle$

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$$a_0 |+0\rangle + a_1 |-1\rangle$$

$$a_0 \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \otimes |0\rangle + a_1 \cdot \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) |1\rangle$$

$$\frac{a_0}{\sqrt{2}} |00\rangle + \frac{a_0}{\sqrt{2}} |10\rangle + \frac{a_1}{\sqrt{2}} |01\rangle - \frac{a_1}{\sqrt{2}} |11\rangle$$

$$|0\rangle |1\rangle \qquad |01\rangle$$

$$|0\rangle \otimes |1\rangle$$

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$\Rightarrow a_0|00\rangle + a_1|11\rangle$

$\Rightarrow$ Outcome = 0: $i \oplus j = 0 \Rightarrow (i=j)$ $a_0|000\rangle + a_1|101\rangle$

$\Rightarrow$ Outcome = 1: $i \oplus j = 1$ $(i \neq j)$ $a_0|01\cancel{1}\rangle + a_1|11\cancel{0}\rangle$ (1 → 0, 0 → 1)

Alice sends the outcome to Bob.
If outcome = 1, then Bob flips his bit.
$a_0|010\rangle + a_1|111\rangle$

Dropping the middle bit: $a_0|00\rangle + a_1|11\rangle$

Alice Bob.

Almost done, except that Alice's qubit is still entangled with Bob's. If she does a measurement then it will destroy Bob's copy.

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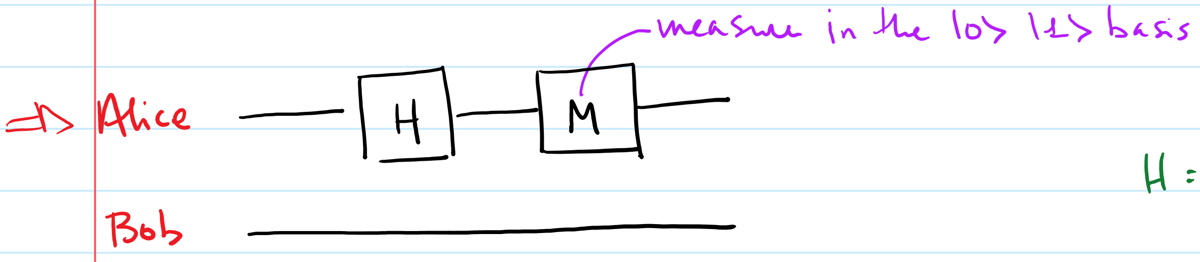
$a_0|\cancel{0}+\rangle + a_1|\cancel{1}-\rangle$

$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$



$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

After H: $a_0|+0\rangle + a_1|-1\rangle$

$$\frac{a_0}{\sqrt{2}}\left(|00\rangle + |10\rangle\right) + \frac{a_1}{\sqrt{2}}\left(|01\rangle - |11\rangle\right)$$

Outcome 0

$a_0|00\rangle + a_1|01\rangle$

$= |0\rangle \otimes \left(a_0|0\rangle + a_1|1\rangle\right)$

$= |0\rangle \otimes |\psi\rangle$

Bob does Nothing

Outcome 1

$a_0|10\rangle - a_1|11\rangle$

$= |1\rangle \otimes \left(a_0|0\rangle - a_1|1\rangle\right)$

Bob does a phase flip

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Alice sends outcome to Bob

$|1\rangle \otimes \left(a_0|0\rangle + a_1|1\rangle\right)$

$= |1\rangle \otimes |\psi\rangle$.

$$\frac{a_0}{\sqrt{2}}|00\rangle + \frac{a_1}{\sqrt{2}}|01\rangle$$

$1/\sqrt{2}$

$$\left[\left(\frac{a_0}{\sqrt{2}}\right)^2 + \left(\frac{a_1}{\sqrt{2}}\right)^2\right]^{1/2}$$

$$\left[\frac{a_0^2}{2} + \frac{a_1^2}{2}\right]^{1/2} = \frac{\left(|a_0|^2 + |a_1|^2\right)^{1/2}}{\sqrt{2}}$$