Quantum Fourier Transform - page 1

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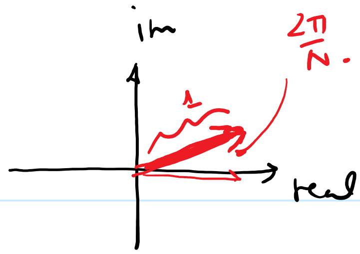

Discrete Fourier Transform.

Input: $(\alpha_0, \dots, \alpha_{N-1}) \in \mathbb{C}^N$.

Output: $(\hat{\alpha}_0, \hat{\alpha}_1, \dots, \hat{\alpha}_{N-1}) \in \mathbb{C}^N$.

$$\hat{\alpha}_k = \sum_{j=0}^{N-1} \omega^{jk} \alpha_j$$

$$\omega = e^{\frac{2\pi i}{N}} = \cos\left(\frac{2\pi}{N}\right) + i \sin\left(\frac{2\pi}{N}\right)$$

The DFT can be seen as matrix multiplication by:

$$\frac{1}{\sqrt{N}} \begin{array}{c|c} & k \\ \hline j & \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)} \\ \vdots & & & & \\ & & & \omega^{jk} & \\ \vdots & & & & \\ 1 & \omega^{N-1} & & & \omega^{(N-1)(N-1)} \end{bmatrix} \end{array} \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_{N-1} \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_0 \\ \vdots \\ \hat{\alpha}_{N-1} \end{bmatrix}$$

DFT_N.

If rows & cols numbered $0 \dots N-1$

$$[DFT]_{jk} = \omega^{jk}$$

Note: $\omega^N = \left[e^{2\pi i/N}\right]^N = \left[e^{2\pi i}\right] = 1$.

$$\overset{1}{\cos(2\pi)} + i \overset{0}{\sin(2\pi)} = 1.$$

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$(\omega^k)^* = \omega^{-k}.$

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$$|f_k\rangle = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^0 \\ \omega^{1\cdot k} \\ \omega^{2\cdot k} \\ \vdots \\ \omega^{(N-1)k} \end{bmatrix}$$

the $|f_k\rangle$'s form an ortho-normal basis of $\mathbb{C}^N$:

$$\langle f_k | f_j \rangle = \frac{1}{N} \sum_{a=0}^{N-1} \omega^{-ak} \cdot \omega^{aj}$$

$$\left(\omega^0\, \omega^{-k}\, \omega^{-2k} \, -- \, \omega^{-k(N-1)}\right) \begin{bmatrix} \omega^0 \\ \omega^j \\ \vdots \end{bmatrix} = \underbrace{\frac{1}{N} \sum_{a=0}^{N-1} \omega^{a(j-k)}}$$

if $j=k$: $\frac{1}{N} \sum_{a=0}^{N-1} \omega^0 = \frac{N}{N} = 1$

if $j \neq k$: $\underbrace{\frac{1}{N} \sum_{a=0}^{N-1} \left[\omega^{(j-k)}\right]^a}$ α

Identity:

$$\boxed{\sum_{a=0}^{N-1} \alpha^a = \frac{\alpha^N - 1}{\alpha - 1}}$$

Numerator: $\omega^{(j-k)N} - 1$

$$e^{i \cdot \frac{2\pi}{N} \cdot N \cdot (j-k)} - 1. = e^{i2\pi(j-k)} - 1$$

ω

$= 1 - 1 = 0.$

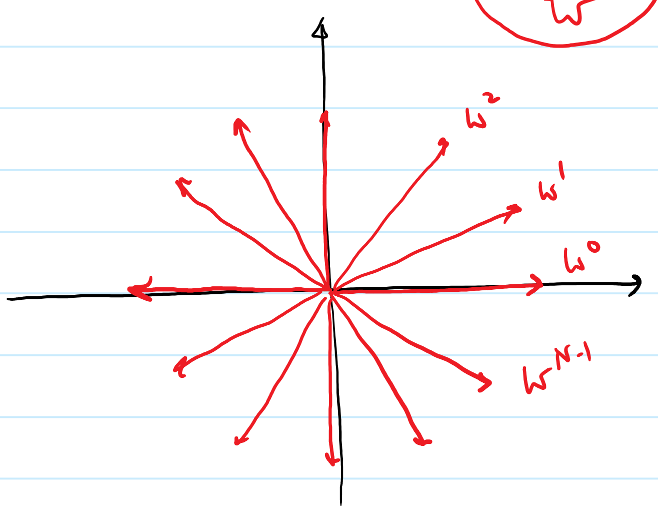

figure shows special case of $j-k=1$
$N = 12.$

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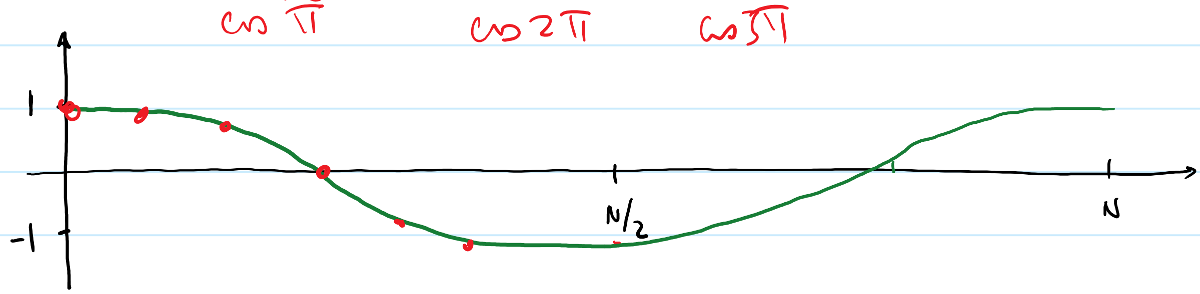
$$\left[W^0 \quad W^k \quad W^{2k} \; \cdots \quad W^{(N-1)k} \right.$$

Real Component:

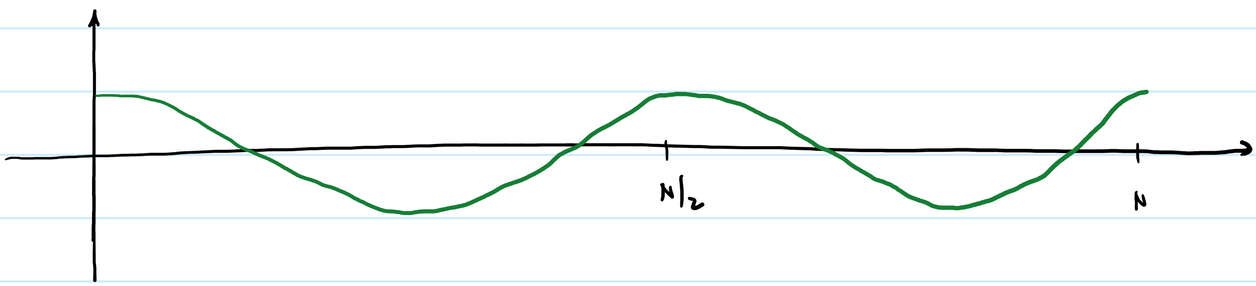
$$\cos(0), \quad \cos \overset{N/2}{k} \cdot \frac{2\pi}{N}, \quad \cos 2k \cdot \frac{2\pi}{N}, \quad \cos 3k \cdot \frac{2\pi}{N}, \; \cdots, \quad \cos (N-1) \cdot k \cdot \frac{2\pi}{N}$$

$\cos \pi \qquad \cos 2\pi \qquad \cos 3\pi$

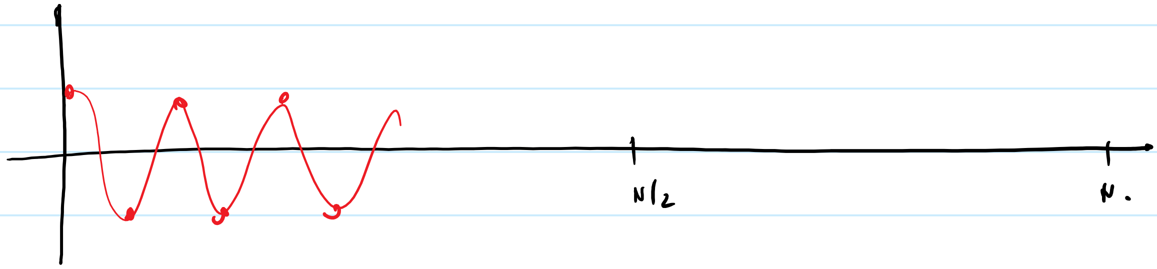
$k=1$



$k=2$



$k = \frac{N}{2}$



$$\cos \quad a \cdot \frac{N}{2} \frac{2\pi}{N} = \cos a\pi. \qquad a = 0 \ldots, N-1.$$

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Early discovery in Quantum Algorithms:

Quantum circuit can calculate the Fourier Transform very efficiently when input vector is encoded in the amplitudes of a quantum state.

This is not a faster way to compute the classical Fourier Transform

- Output vector only accessible via quantum measurement.

Still the QFT (Quantum Fourier Transform) important component in many quantum algorithms.

Back to the classical DFT before we get to the QFT.

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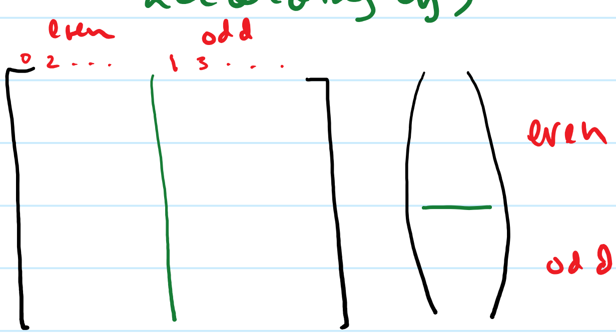
Computing the DFT is matrix multiplication:

$$\frac{1}{\sqrt{N}}\begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & \omega & \omega^2 & \cdots & \omega^{N-1}\\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(N-1)}\\ \vdots & & & & \\ 1 & \omega^{N-1} & & & \omega^{(N-1)(N-1)}\end{bmatrix}\begin{bmatrix}\alpha_0\\ \vdots\\ \alpha_{N-1}\end{bmatrix} = \begin{bmatrix}\hat{\alpha}_0\\ \vdots\\ \hat{\alpha}_{N-1}\end{bmatrix}$$

$O(N^2)$ multiplications.

Fast Fourier Transform (FFT) exploits the structure in the matrix to compute the DFT more efficiently. → Assume $N = 2^n$ $n \in \mathbb{Z}^+$.

Reorganize columns of the matrix so that columns with an even index appear before the columns with an odd index.
(Entries of the input vector must be re-organized according by)



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$\rightarrow \frac{DFT_{N/2}}{\sqrt{2}}.$

$DFT_N.$

$0 \le k \le \frac{N}{2}-1$

$0 \le j \le \frac{N}{2}-1$

0
1
2
3
4

5+0
5+1
5+2
5+3
5+4

$$
\begin{array}{c|cc}
 & 2k & 2k+1 \\
\hline
j & \dfrac{w^{j\cdot 2k} = (w^2)^{j\cdot k}}{\sqrt{N}} & \dfrac{w^{j(2k+1)} = (w^2)^{jk}\cdot w^j}{\sqrt{N}} \\
j+\frac{N}{2} & \dfrac{w^{(j+\frac{N}{2})2k}}{\sqrt{N}} & \dfrac{w^{(j+\frac{N}{2})(2k+1)}}{\sqrt{N}}
\end{array}
$$

Define $\bar{w} = w^2$

$0 \le k \le \frac{N}{2}-1$

$0 \le j \le \frac{N}{2}-1$

$$
\frac{1}{\sqrt{N}}
\begin{bmatrix}
\bar{w}^0 & \bar{w}^0 & \cdots & \bar{w}^0 \\
\bar{w}^0 & \bar{w}^1 & \cdots & \bar{w}^{\frac{N}{2}-1} \\
\vdots & & & \\
\bar{w}^0 & \bar{w}^{\frac{N}{2}-1} & \cdots & \bar{w}^{(\frac{N}{2}-1)(\frac{N}{2}-1)}
\end{bmatrix}
$$

$\bar{w}^{j\cdot k}.$

$$\bar{w} = w^2 = e^{\frac{2\pi i}{N}\cdot 2} = e^{\frac{2\pi i}{(N/2)}}$$

$$
DFT_{N/2} = \frac{1}{\sqrt{N/2}}
\begin{bmatrix}
\bar{w}^0 & \bar{w}^0 & \cdots & \bar{w}^0 \\
\bar{w}^0 & \bar{w}^1 & \cdots & \bar{w}^{\frac{N}{2}-1} \\
\vdots & & & \\
\bar{w}^0 & \bar{w}^{\frac{N}{2}-1} & \cdots & \bar{w}^{(\frac{N}{2}-1)(\frac{N}{2}-1)}
\end{bmatrix}
$$

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$\frac{1}{\sqrt2} W_{\frac N2} \cdot DFT_{N/2}$

$\frac{1}{\sqrt2} DFT_{N/2}$

$$\begin{array}{c|cc}
 & 2k & 2k+1 \\
\hline
j & \frac{w^{j\cdot 2k}}{\sqrt N} = \frac{(w^2)^{j\cdot k}}{\sqrt N} & \frac{w^{j(2k+1)}}{\sqrt N} = \frac{(w^2)^{jk}\cdot w^{j}}{\sqrt N} \\
j+\frac N2 & \frac{w^{(j+\frac N2)2k}}{\sqrt N} & \frac{w^{(j+\frac N2)(2k+1)}}{\sqrt N}
\end{array} = \frac{1}{\sqrt2} W_{N/2} \cdot DFT_{N/2}.$$

$\frac{1}{\sqrt2} DFT_{N/2}$

$0 \le k \le \frac N2 - 1$

$0 \le j \le \frac N2 - 1$

This submatrix is: $\frac{1}{\sqrt2} DFT_{N/2}$

$$\left(w^{\frac N2 \cdot 2}\right)^k = \left(w^N\right)^k$$

$$w^{(j+\frac N2)2k} = w^{2kj} \cdot \underbrace{\cancel{w^{N/2 \cdot 2k}}}_{1} = w^{2kj}$$

$$w^{j(2k+1)} = w^{2kj}\, w^{j}$$

$$\left(e^{\frac{2\pi i}{N}}\right)^{N/2} = e^{i\pi}$$

Define $W_{\frac N2} = \begin{bmatrix} w^0 & & & & 0 \\ & w^1 & & & \\ & & w^2 & & \\ & & & \ddots & \\ 0 & & & & w^{N/2-1} \end{bmatrix}$

Note using $w = e^{2\pi i/N}$.

$$w^{(j+\frac N2)(2k+1)} = w^{2jk} \cdot \cancel{w^{\frac N2 \cdot 2k}} \cdot w^{j} \cdot \underbrace{w^{N/2}}_{-1}$$

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$$
\begin{array}{c}
j \\[2em] \frac{N}{2}+j
\end{array}
\begin{bmatrix}
\frac{DFT_{N/2}}{\sqrt{2}} & \frac{W_{N/2}\, DFT_{N/2}}{\sqrt{2}} \\[1em]
\frac{DFT_{N/2}}{\sqrt{2}} & -\frac{W_{N/2}\, DFT_{N/2}}{\sqrt{2}}
\end{bmatrix}
\begin{pmatrix}
\alpha_0 \\ \alpha_2 \\ \vdots \\ \alpha_{N-2} \\ \alpha_1 \\ \alpha_3 \\ \vdots \\ \alpha_{N-1}
\end{pmatrix}
\begin{array}{l}
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \vec{\alpha}_e \\
\left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \vec{\alpha}_o.
\end{array}
=
\begin{pmatrix}
\hat{\alpha}_0 \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \vdots \\ \hat{\alpha}_{N-1}
\end{pmatrix}
\begin{array}{l} \} \hat{\alpha}_j \\ \\ \\ \\ \end{array}
$$

ω^j

$(DFT_{N/2})_j \triangleq$ row j of $(DFT_{N/2})$

For $0 \le j \le \frac{N}{2}-1$

$$\hat{\alpha}_j = \frac{1}{\sqrt{2}}\left[(DFT_{N/2})_j \cdot \vec{\alpha}_e + \omega^j (DFT_{N/2})_j \cdot \vec{\alpha}_o\right]$$

$$\hat{\alpha}_{j+\frac{N}{2}} = \frac{1}{\sqrt{2}}\left[(DFT_{N/2})_j \cdot \vec{\alpha}_e - \omega^j (DFT_{N/2})_j \cdot \vec{\alpha}_o\right]$$

$\frac{1}{\sqrt{N/2}}$

$$\hat{\alpha}_j = \frac{1}{\sqrt{2}}\left(\vec{\alpha}_e\, (DFT_{N/2})_j + \left[\vec{\alpha}_o\, (DFT_{N/2})_j\right] \omega^j\right)$$

$$\hat{\alpha}_{j+\frac{N}{2}} = \frac{1}{\sqrt{2}}\left(\vec{\alpha}_e\, (DFT_{N/2})_j - \left[\vec{\alpha}_o\, (DFT_{N/2})_j\right] \cdot \omega^j\right)$$

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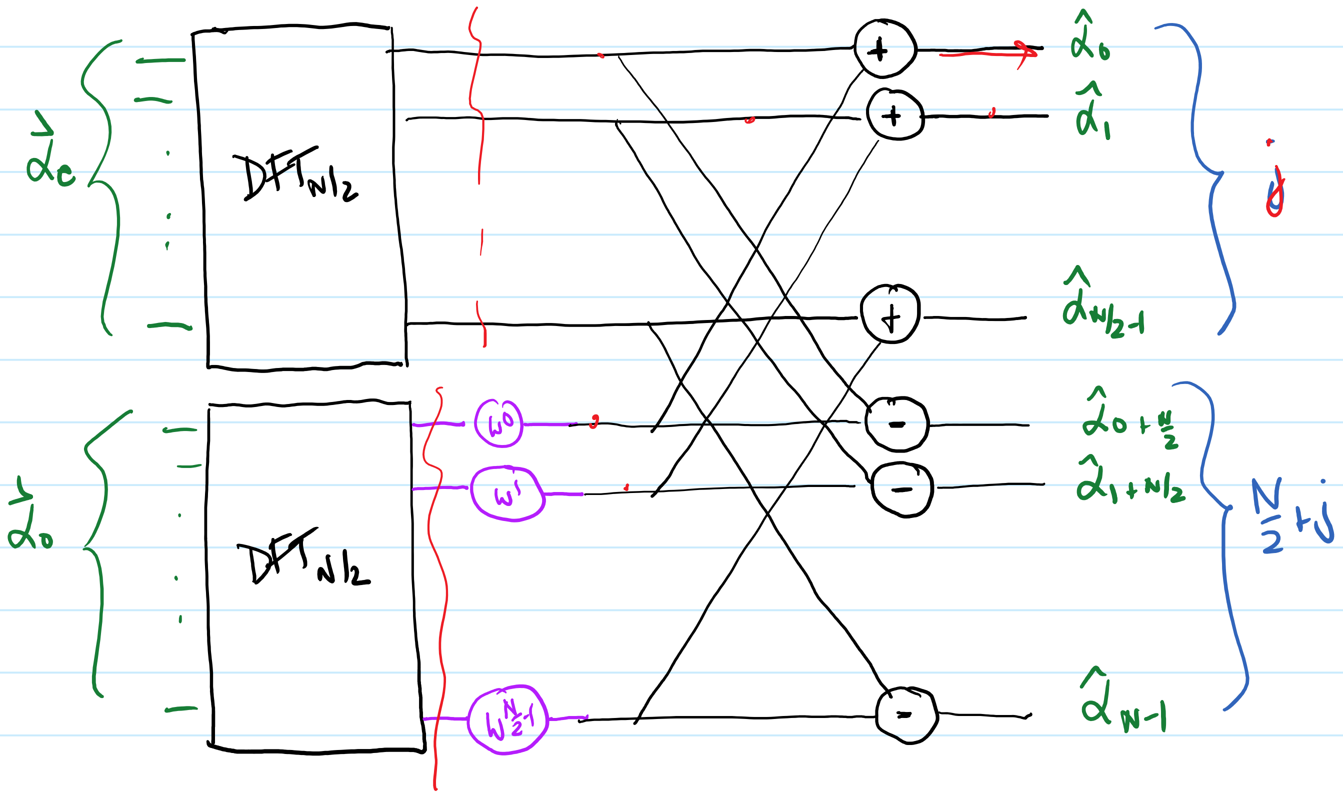
For $0 \le j \le \frac{N}{2}-1$

$$\hat{\alpha}_j = \frac{1}{\sqrt{2}} (DFT_{N/2})_j \cdot \vec{\alpha}_e + \frac{1}{\sqrt{2}} W^j (DFT_{N/2})_j \cdot \vec{\alpha}_o$$

$$\hat{\alpha}_{\frac{N}{2}+j} = \frac{1}{\sqrt{2}} (DFT_{N/2})_j \cdot \vec{\alpha}_e - \frac{1}{\sqrt{2}} W^j (DFT_{N/2})_j \cdot \vec{\alpha}_o$$



Def:

$x, y \to (+) \to \frac{1}{\sqrt{2}}(x+y)$ $\qquad$ $x, y \to (-) \to \frac{1}{\sqrt{2}}(x-y)$

Complexity for input of size N is $S(N)$

$$S(N) = 2 \cdot S(N/2) + O(N) \longrightarrow S(N) = O(N \log N)$$

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Quantum Fourier Transform

$$\underbrace{QFT}_{N \times N \text{ matrix}} |\psi\rangle = |\hat{\psi}\rangle$$

$$|\psi\rangle = \sum_{j=0}^{N-1} \alpha_j |j\rangle$$

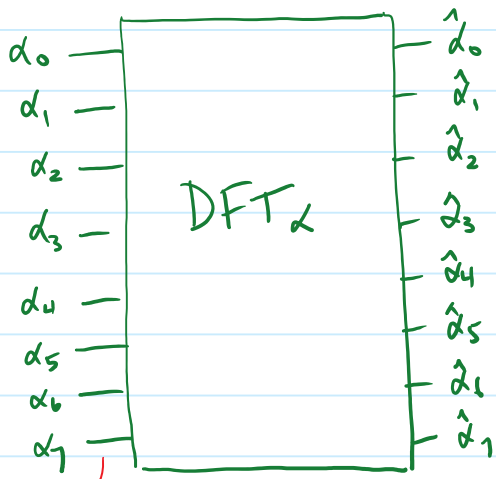
if N is 2^n then j can be represented as an n-bit string.

$$|\hat{\psi}\rangle = \sum_{j=0}^{N-1} \hat{\alpha}_j |j\rangle$$

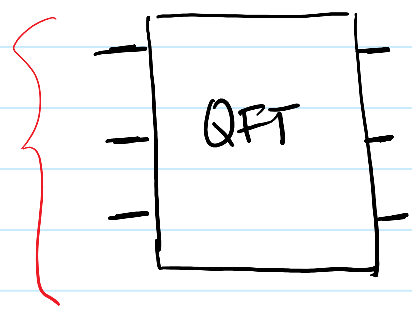
$$\hat{\alpha}_k = \sum_{j=0}^{N-1} \omega^{jk} \alpha_j \qquad \omega \triangleq e^{\frac{2\pi i}{N}}$$

Example $N = 8$ $n = 3$

Classical "Circuit"



Each line represents a register that can hold a ~~real~~ complex number.



Input $|\psi\rangle$ is a superposition over 8 standard basis states. $|000\rangle, \ldots, |111\rangle$

α_j is amplitude of $|j\rangle$

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QFT_N is an $N \times N$ matrix $(N = 2^n)$.

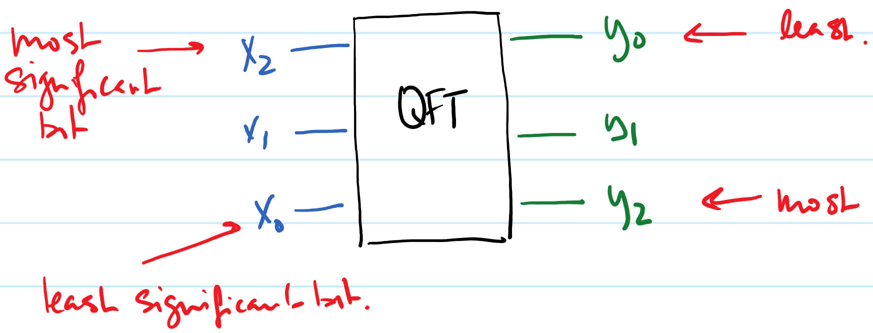
Computed by a quantum circuit with n input + output qubits.

$QFT_N |\psi\rangle = |\hat{\psi}\rangle$ unitary.

this is exactly the DFT_N matrix

$x = x_2 x_1 x_0$

Our circuit for QFT will reverse the output string:



$$\text{Output } |\hat{\psi}\rangle = \sum_{j \in \{0,1\}^n} \alpha_j \, |j\text{-rev}\rangle$$

$$\alpha_{000}|000\rangle + \alpha_{001}|100\rangle + \alpha_{010}|010\rangle + \alpha_{011}|110\rangle + \alpha_{100}|001\rangle$$
$$+ \alpha_{101}|101\rangle + \alpha_{110}|011\rangle + \alpha_{111}|111\rangle$$

Can be corrected at the very end by SWAP gates.

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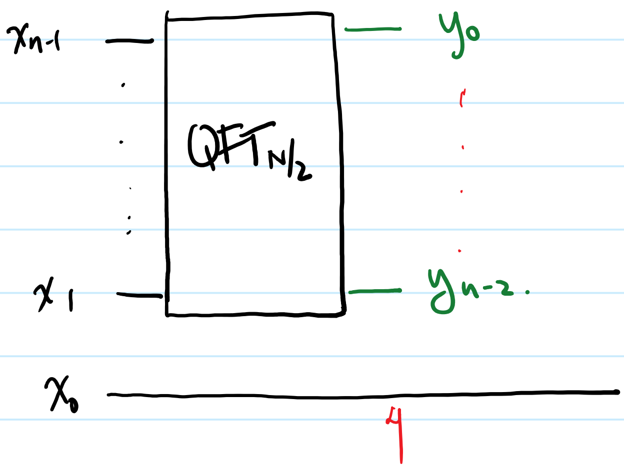
Use the least significant bit to separate between odd and even numbers:

Perform $QFT_{N/2}$ on the $(n-1)$ most significant bits:



$x \in \{0,1\}^n$

$x = x_{n-1} x_{n-2} \cdots x_1 x_0$

0 if x is even
1 if x is odd.

$$|\psi\rangle = \sum_{j \in \{0,1\}^n} \alpha_j |j\rangle = \sum_{\substack{j\ \text{even} \\ \text{last bit} = 0}} \alpha_j |j\rangle + \sum_{\substack{j\ \text{odd} \\ \text{last bit} = 1}} \alpha_j |j\rangle$$

$$= \underbrace{\sum_{j \in \{0,1\}^{n-1}} \alpha_{j0} |j0\rangle}_{\vec{\alpha}_e} + \underbrace{\sum_{j \in \{0,1\}^{n-1}} \alpha_{j1} |j1\rangle}_{\vec{\alpha}_o}$$

$$= \left(\sum_{j \in \{0,1\}^{n-1}} \alpha_{j0} |j\rangle \right) \otimes |0\rangle + \left(\sum_{j \in \{0,1\}^{n-1}} \alpha_{j1} |j\rangle \right) \otimes |1\rangle$$

We have

$$QFT_{N/2} \left(\sum_{j \in \{0,1\}^{n-1}} \alpha_{j0} |j\rangle \right) \otimes |0\rangle + QFT_{N/2} \left(\sum_{j \in \{0,1\}^{n-1}} \alpha_{j1} |j\rangle \right) \otimes |1\rangle$$

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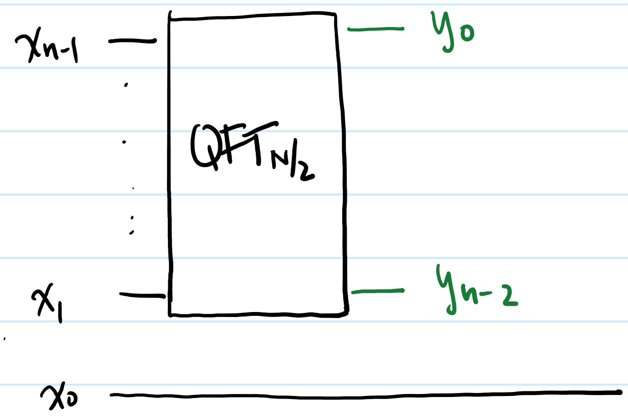

$$QFT_{N/2}\Big(\underbrace{\sum_{j\in\{0,1\}^{n-1}} \alpha_{j0}|j\rangle}_{\vec{\alpha}_e}\Big)|0\rangle + QFT_{N/2}\Big(\sum_{j\in\{0,1\}^{n-1}} \alpha_{j1}|j\rangle\Big)|1\rangle$$

Want:

$$QFT_{N/2}\Big(\underbrace{\sum_{j\in\{0,1\}^{n-1}} \alpha_{j0}|j\rangle}_{\vec{\alpha}_e}\Big)|0\rangle + W_{\frac{N}{2}}\, QFT_{N/2}\Big(\underbrace{\sum_{j\in\{0,1\}^{n-1}} \alpha_{j1}|j\rangle}_{\vec{\alpha}_o}\Big)|1\rangle$$

Need to implement the following operation.

$$|j\rangle|0\rangle \longrightarrow |j\rangle|0\rangle$$

$$|j\rangle|1\rangle \longrightarrow \omega^j|j\rangle|1\rangle$$

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Need to implement the following operation.

$$|j\rangle|0\rangle \longrightarrow |j\rangle|0\rangle$$

$$|j\rangle|1\rangle \longrightarrow \omega^{j}|j\rangle|1\rangle$$

Example $j = 13$. binary rep: $1\ 1\ 0\ 1$ ($j_3\ j_2\ j_1\ j_0$)

$$13 = 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0$$

$$\omega^{13} = \omega^{1\cdot 2^3} \cdot \omega^{1\cdot 2^2} \cdot \omega^{0\cdot 2^1} \cdot \omega^{1\cdot 2^0}$$

$$= \omega^{2^3} \cdot \omega^{2^2} \cdot \omega^{2^0}$$

ω^j

For $k = 0$ to $n-2$

if (k^{th} bit of j $= 1$) and ($x_{n-1} = 1$)

multiply by ω^{2^k}

k^{th} least significant bit.

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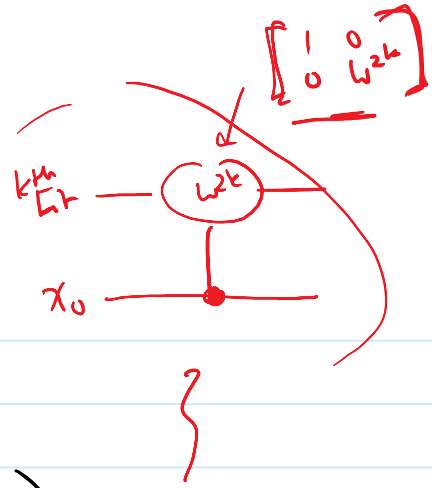


for $k = 0$ to $n-2$

if (k^{th} bit $= 1$) and ($X_0 = 1$)

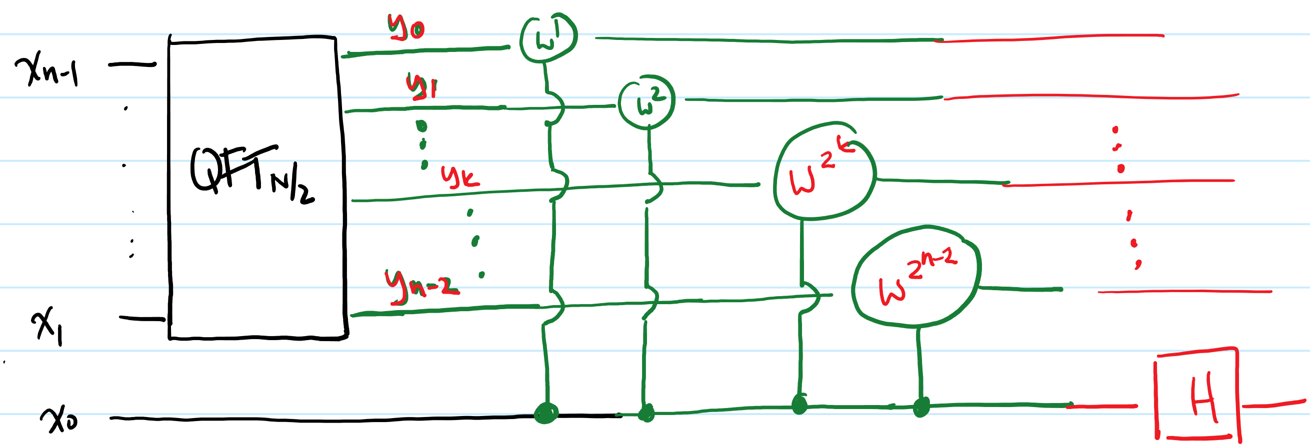
k^{th} least significant bit.

multiply by w^{2^k}

$$\begin{array}{c} \\ \end{array}\begin{array}{cccc} 00 & 01 & 10 & 11 \end{array}$$

$$\begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & 1 & \\ 0 & & & w^{2^k} \end{bmatrix}$$

k^{th} bit, x_0



$$QFT_{N/2}\Big(\overbrace{\sum_{j\in\{0,1\}^{n-1}} \alpha_{j0}\,|j\rangle}^{\vec{\alpha}_e}\Big)|0\rangle + W_{\frac{N}{2}}\, QFT_{N/2}\Big(\overbrace{\sum_{j\in\{0,1\}^{n-1}} \alpha_{j1}\,|j\rangle}^{\vec{\alpha}_o}\Big)|1\rangle$$

Now apply final H:

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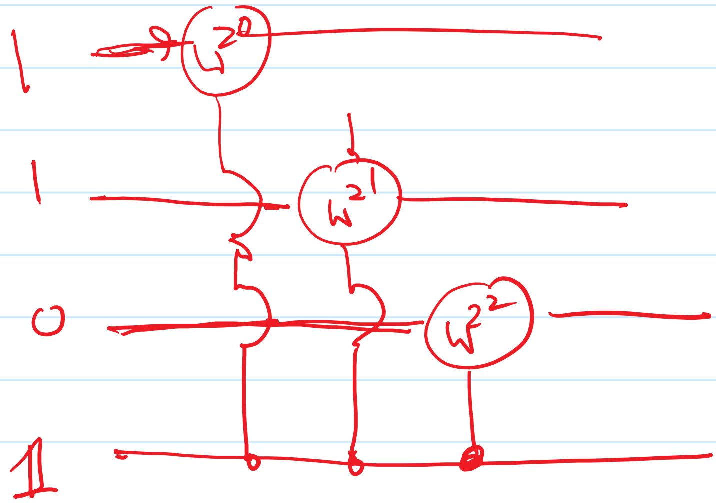
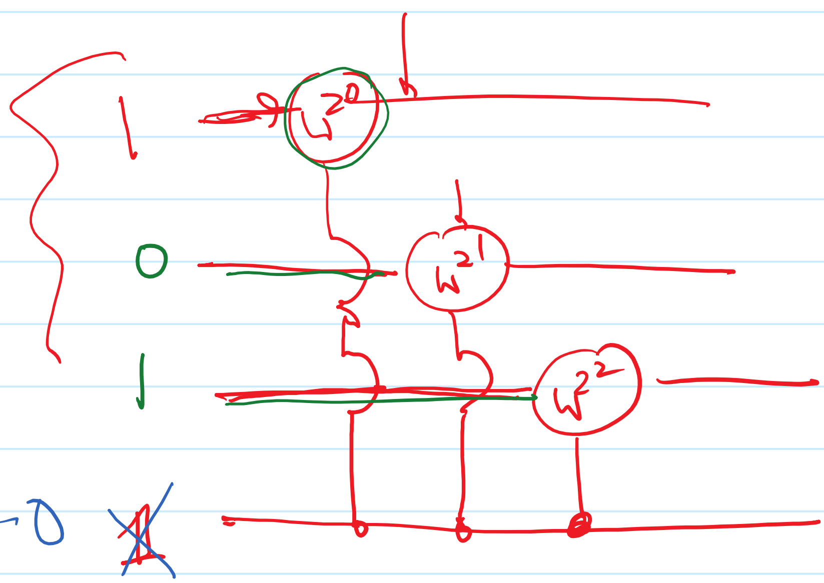
Let's see what these gates do for n=3:

the string 011 = 3 in binary



$|1101\rangle \rightarrow \omega^2 \cdot \omega |1101\rangle = \omega^3 |1101\rangle$

the string "101" represents 5.



$|1011\rangle \rightarrow \omega^{2^0} \cdot \omega^{2^2} |1011\rangle = \omega^5 |1011\rangle$

if last bit = 0, then $|1011\rangle \rightarrow |1011\rangle$

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Input will actually be a quantum state $\underbrace{|\phi_0\rangle}_{3\text{ qubit}}|0\rangle + \underbrace{|\phi_1\rangle}_{3\text{ qubit.}}|1\rangle$

$$|\phi_1\rangle = \alpha_0|000\rangle + \alpha_1|100\rangle + \alpha_2|010\rangle + \alpha_3|110\rangle$$
$$\alpha_4|001\rangle + \alpha_5|101\rangle + \alpha_6|011\rangle + \alpha_7|111\rangle$$

(note that strings are reversed).

The circuit will change the state to:

$$|\phi_0\rangle|0\rangle +$$

no change if last bit = 0.

$$\left(\alpha_0|000\rangle + \omega\cdot\alpha_1|100\rangle + \omega^2\alpha_2|010\rangle + \omega^3\alpha_3|110\rangle + \alpha_4\omega^4|001\rangle + \alpha_5\omega^5|101\rangle + \alpha_6\omega^6|011\rangle + \alpha_7\omega^7|111\rangle\right)$$
$$\otimes|1\rangle$$

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Now apply final H:

$$\mathrm{QFT}_{N/2}\Big(\underbrace{\sum_{j\in\{0,1\}^{n-1}} \alpha_{j0}|j\rangle}_{\vec{\alpha}_e}\Big)\overbrace{|0\rangle}^{H} + W_{N/2}\,\mathrm{QFT}_{N/2}\Big(\underbrace{\sum_{j\in\{0,1\}^{n-1}} \alpha_{j1}|j\rangle}_{\vec{\alpha}_o}\Big)|1\rangle$$

$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\frac{1}{\sqrt{2}}\mathrm{QFT}_{N/2}\Big(\sum_{j\in\{0,1\}^{n-1}} \alpha_{j0}|j\rangle\Big)|0\rangle + \frac{1}{\sqrt{2}} W_{\frac{N}{2}}\,\mathrm{QFT}\Big(\sum_{j\in\{0,1\}^{n-1}} \alpha_{j1}|j\rangle\Big)|0\rangle$$

$$+\frac{1}{\sqrt{2}}\mathrm{QFT}_{N/2}\Big(\sum_{j\in\{0,1\}^{n-1}} \alpha_{j0}|j\rangle\Big)|1\rangle - \frac{1}{\sqrt{2}} W_{\frac{N}{2}}\,\mathrm{QFT}\Big(\sum_{j\in\{0,1\}^{n-1}} \alpha_{j1}|j\rangle\Big)|1\rangle$$

The last bit becomes the most significant bit.
(This is why the bits are reversed in the output).

Complexity: $S(n) = S(n-1) + O(n)$

Solve recurrence: $S(n) = O(n^2)$.

$\Rightarrow S(\log^2 N)$.

Quantum Fourier Transform - page 16

Friday, November 2, 2018 9:57 AM

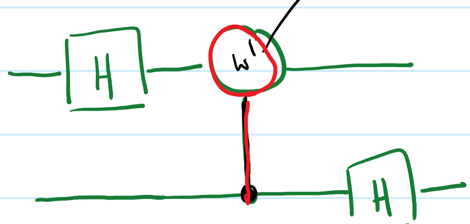
$N=2 \quad n=1 \qquad \omega = e^{\frac{2\pi i}{2}} = e^{\pi i} = -1.$

$$\frac{1}{\sqrt{N}}\begin{bmatrix} \omega^0 & \omega^0 \\ \omega^0 & \omega^1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$$

$N=4 \quad n=2. \qquad \omega = e^{\frac{2\pi i}{4}} = e^{\frac{\pi}{2}i} = i$

Controlled: $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

$$\underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}}_{\text{this is the full Unitary matrix}}$$



this is the 2-qubit quantum circuit.

Does the circuit really compute the operation specified in the matrix?

Input State: $\alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$

After controlled ω: $\frac{\alpha_0+\alpha_2}{\sqrt{2}}|00\rangle + \frac{\alpha_0-\alpha_2}{\sqrt{2}}|10\rangle + \frac{\alpha_1+\alpha_3}{\sqrt{2}}|01\rangle + i\frac{\alpha_1-\alpha_3}{\sqrt{2}}|11\rangle$

Then apply H to second qubit and verify the result is:

$$\frac{1}{2}(\alpha_0+\alpha_1+\alpha_2+\alpha_3)|00\rangle + \frac{1}{2}(\alpha_0+i\alpha_1-\alpha_2-i\alpha_3)|10\rangle$$
$$+\frac{1}{2}(\alpha_0-\alpha_1+\alpha_2-\alpha_3)|01\rangle + \frac{1}{2}(\alpha_0-i\alpha_1-\alpha_2+i\alpha_3)|11\rangle.$$