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Postulates of Quantum Mechanics.

Quantum Mechanics is a mathematical framework for describing nature.

Not a description of nature on its own.

Means of describing:

- States of a physical system.
- How states change over time.
- What information can be extracted from the system.

Superposition:

Classical World:

Deterministic: bit is 0 or 1.

L_1 norm of $(x_1, \ldots, x_n) = \sum_{i=1}^{n} |x_i|$

Probabilistic:

Value of the bit is unknown but can be seen as a probability distribution

p_0 = prob the bit is 0

p_1 = prob the bit is 1

$p_0, p_1 \geq 0$

$p_0 + p_1 = 1.$

State (p_0, p_1) → L_1 norm $= 1$.

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Quantum bit (qubit)

Simultaneously partially 0 + partially 1

"amplitude" (instead of probability)

α_0 - amplitude for 0

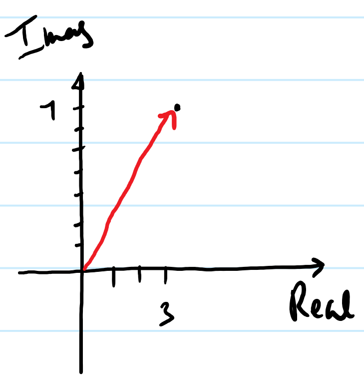
α_1 - amplitude for 1

$\alpha_0, \alpha_1 \in \mathbb{C}$ (complex numbers)

State: $\alpha_0 |0\rangle + \alpha_1 |1\rangle$ → Dirac notation.

Complex Number Review: $\alpha \in \mathbb{C}$

$\alpha = \underbrace{a}_{\text{Real}} + \underbrace{bi}_{\text{Imaginary}}$ $i = \sqrt{-1}$ $a, b \in \mathbb{R}$.



$\alpha = 3 + 7i$

$\beta = 4 - 5i$

$\alpha + \beta = 7 + 2i$

$\alpha \cdot \beta = (3+7i)(4-5i)$

$= 12 + 35 + 13i$

$= 47 + 13i$

$|\alpha| = \sqrt{58}$

$|\alpha|^2 = 3^2 + 7^2 = 58$

Note $|\alpha| \geq 0$ and $|\alpha| = 0$ iff $\alpha = 0$.

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$(\alpha+\beta)^* = \alpha^* + \beta^*$

$(\alpha\beta)^* = \alpha^*\beta^*$

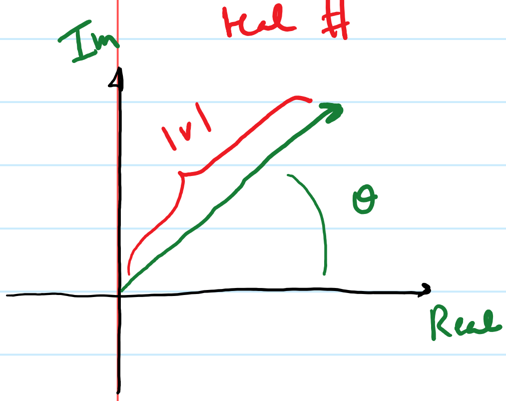
$\alpha = a+bi$ $\alpha^* = a-bi$ (conjugate of α).

$\alpha \cdot \alpha^* = \alpha^* \cdot \alpha = (a+bi)(a-bi)$

$= a^2+b^2 = |\alpha|^2$

"Phasor" representation of a complex number:

$$\underbrace{|v|}_{\text{magnitude is a non-negative real \#}} \underbrace{e^{i\theta}}_{\text{"phase"}} = |v|\cos\theta + i|v|\sin\theta$$



Express: $5 \cdot e^{i\frac{3\pi}{4}}$ in standard form.

$5 \cdot \cos\left(\frac{3\pi}{4}\right) + i\, 5 \cdot \sin\left(\frac{3\pi}{4}\right)$

$-\frac{5}{\sqrt{2}} + i\frac{5}{\sqrt{2}}$

$\alpha = 5 \cdot e^{i3\pi/4}$ $\alpha^* = 5 \cdot e^{-i3\pi/4}$

$|\alpha|^2 = \alpha\alpha^* = 5 \cdot 5 \cdot e^{i3\pi/4} \cdot e^{-i3\pi/4} = 25$

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$\alpha = a+bi$ $\qquad \alpha^* = a - bi$ (conjugate of α).

$\alpha \cdot \alpha^* = \alpha^* \cdot \alpha = (a+bi)(a-bi)$
$= a^2+b^2 = |\alpha|^2$

State of a qubit: $\alpha_0|0\rangle + \alpha_1|1\rangle$

What does a negative or complex amplitude mean?

If we measure the value of the qubit
$\alpha_0|0\rangle + \alpha_1|1\rangle$

With probability $|\alpha_0|^2$

Outcome of measurement is $|0\rangle$
New state (after measurement) is $|0\rangle$

With probability $|\alpha_1|^2$

Out come is $|1\rangle$
New state is $|1\rangle$

Requirement:
$|\alpha_0|^2 + |\alpha_1|^2 = 1$

L_2 norm of
(α_0, α_1) is 1.

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What about n qubits?

Set of all n-bit strings.

Classical bits: state $= x \in \{0,1\}^n$

Probabilistic classical state:

p_x = probability of state x.

State: $(p_0, p_1, \ldots, p_{2^n-1})$ ← L_1 norm of entries $= 1$.

Quantum State:

$$\sum_{x \in \{0,1\}^n} \alpha_x |x\rangle$$

$$= \alpha_{0\ldots0}|0\ldots0\rangle + \alpha_{0\ldots1}|0\ldots01\rangle \ldots \quad \alpha_{1\ldots1}|11\ldots1\rangle$$

Corresponds to vector $(\alpha_0, \alpha_1, \ldots, \alpha_{2^n-1})$

Need $\sum_x |\alpha_x|^2 = 1$. (L_2-norm $= 1$).

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Example: 3 qubits.

$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle + \alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

$$(\alpha_{000}, \alpha_{001}, \alpha_{010}, \alpha_{011}, \alpha_{100}, \alpha_{101}, \alpha_{110}, \alpha_{111})$$

Measure all 3 qubits:

Outcome is 010 with probability $|\alpha_{010}|^2$.
Afterwards state is $|010\rangle$

$$(0, 0, \overset{010}{1}, 0, 0, 0, 0, 0)$$

If we measure the qubits again, the outcome is 010 with probability 1.

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$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle$$
$$+ \alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

Measure first qubit only:

Prob outcome is 0 $= p_0 = |\alpha_{000}|^2 + |\alpha_{001}|^2 + |\alpha_{010}|^2 + |\alpha_{011}|^2$

After wards state collapses to:

$$\frac{\alpha_{000}}{\sqrt{p_0}}|000\rangle + \frac{\alpha_{001}}{\sqrt{p_0}}|001\rangle + \frac{\alpha_{010}}{\sqrt{p_0}}|010\rangle + \frac{\alpha_{011}}{\sqrt{p_0}}|011\rangle$$

$$\left(\alpha_{000}/\sqrt{p_0},\ \alpha_{001}/\sqrt{p_0},\ \alpha_{010}/\sqrt{p_0},\ \alpha_{100}/\sqrt{p_0},\ 0, 0, 0, 0\right)$$

Check that L_2- norm still $= 1$:

$$\frac{\alpha_{000}^* \alpha_{000}}{\sqrt{p_0}\sqrt{p_0}} + \frac{\alpha_{001}^* \cdot \alpha_{001}}{p_0} + \frac{\alpha_{010}^* \alpha_{010}}{p_0} + \frac{\alpha_{011}^* \alpha_{011}}{p_0}$$

$$= \frac{1}{p_0}\Big(\underbrace{|\alpha_{000}|^2 + |\alpha_{001}|^2 + |\alpha_{010}|^2 + |\alpha_{011}|^2}_{= p_0}\Big)$$

$$= \frac{p_0}{p_0} = 1.$$

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Postulates of Quantum Mechanics: Evolution.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} \rightsquigarrow \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}$$

Quantum State

Need to manipulate data to compute.

Change in state must preserve the l_2-norm!

Transformations must be linear:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{T} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{T} \alpha \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \beta \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Any linear l_2-norm-preserving transformation is unitary : Multiplication by a unitary matrix

to be defined soon.

$$\begin{bmatrix} U \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix}$$

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Using A Quantum Mechanical System to Compute:

1. Storing Information (state vector)
2. Manipulating Information (unitary transformations).
3. Extracting a result (quantum measurement)

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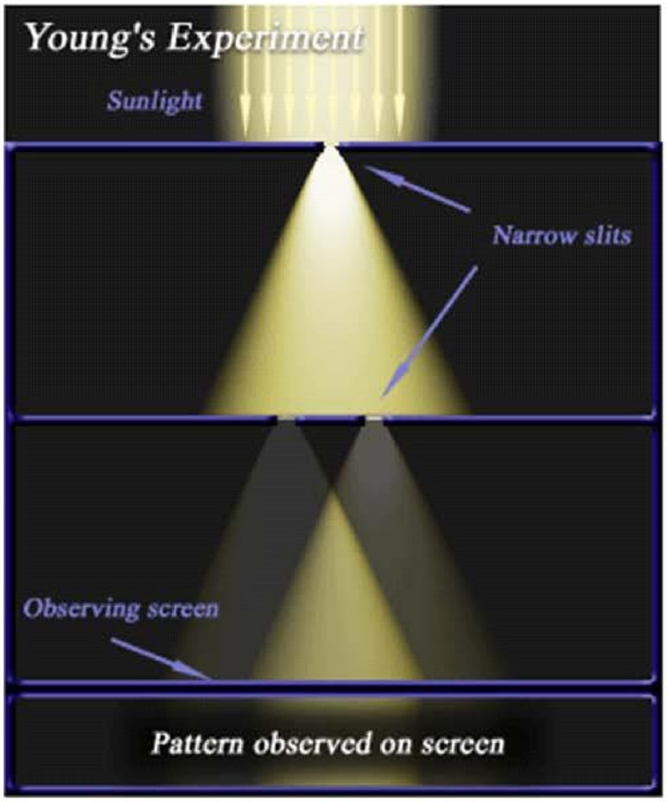

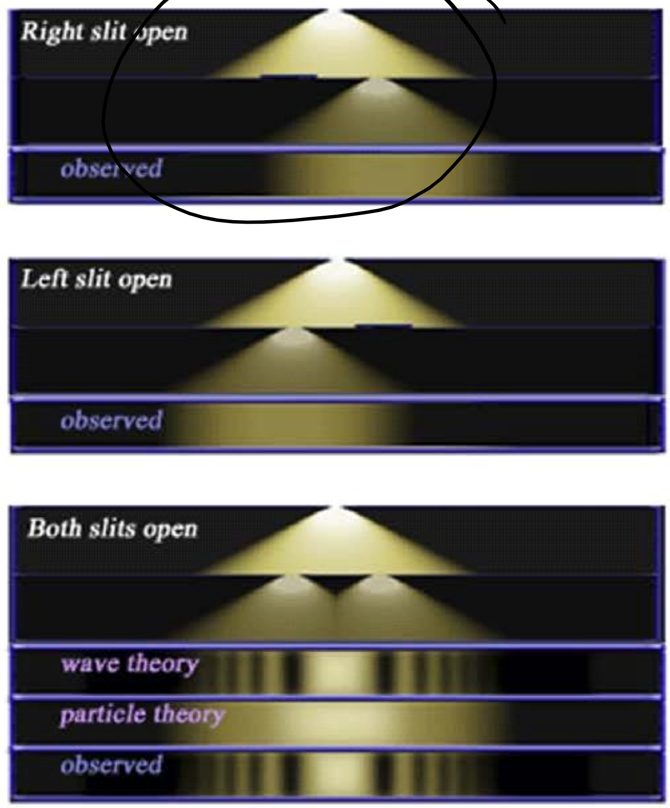

Wave pattern formed by one photon at a time
(no interference between different photons.).

Classical Explanation:

$$p(x) = (\text{prob slit 1})\cdot p(x|\text{slit 1}) + (\text{prob slit 2})\cdot p(x|\text{slit 2})$$

(weighted average of patterns from single slit cases).

Quantum Explanation:

$$\psi(x) = \alpha_1 \underbrace{\psi_1(x)}_{\text{state from slit 1}} + \alpha_2 \underbrace{\psi(x_2)}_{\text{state from slit 2.}}$$

can have negative interference.

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Postulates of Quantum Mechanics:

why complex numbers?

(why not just negative?)

Consider two qubit states.

$\theta = 0$ $\quad\quad\quad\quad$ $\theta = \pi$

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \qquad |-\rangle = \frac{1}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

These two states have the same outcome if measuring whether the qubit is $|0\rangle$ or $|1\rangle$.

Continuous transition from $|+\rangle$ to $|-\rangle$

Intermediate state of form:

$$\frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle \qquad \theta: 0 \to \pi.$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\cdot 0} = \cos 0 + i\sin 0 = 1.$$

$$e^{i\cdot\pi} = \cos\pi + i\sin\pi = -1$$

Measure $|0\rangle$ or $|1\rangle$. Prob of $0 = \left(\frac{1}{\sqrt{2}}\right)^2$

$$\text{Prob of } 1 = \left|\frac{e^{i\theta}}{\sqrt{2}}\right|^2 = \left(\frac{e^{i\theta}}{\sqrt{2}}\right)^*\left(\frac{e^{i\theta}}{\sqrt{2}}\right) = \frac{e^{-i\theta}\cdot e^{i\theta}}{\sqrt{2}\cdot\sqrt{2}} = \frac{1}{2}$$

$$(\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta) = \cos^2\theta + \sin^2\theta = 1.$$