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[due to Kitaev 1995].

Phase estimation is another algorithm that makes essential use of the QFT.

We will use phase estimation for approximate counting: given function f, determine the number of distinct inputs x such that $f(x)=1$.

Phase estimation is used in many other quantum algorithms: especially in chemistry and physics simulations.

eigenvalue

Phase Estimation:
Suppose we have a unitary U with eigenvector $|u\rangle$ and eigenvalue $e^{2\pi i \varphi}$ where φ is unknown.
The goal is to estimate φ. (Actually, we will just estimate $|\varphi|$)

We are given $|u\rangle$ and a set of black boxes that can compute controlled U^{2^j} for non-negative j.
(The actual existence of these black boxes depends on the application.)

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$\langle \phi | \phi' \rangle = \langle \phi \underbrace{U^\dagger \; U}_{} \phi' \rangle$

$U^\dagger U = I$

$U^\dagger = U^{-1}$

Why are the eigenvalues of a unitary operator U always of the $e^{i\theta}$?

If U is unitary then $U^\dagger = U^{-1}$

$$U = \sum_{i=1}^{N} \lambda_i |v_i\rangle\langle v_i|$$

$|v_i\rangle$ eigenvectors of U

λ_i eigenvalues of U.

$1/\lambda_i$

$$U^{-1} = \sum_{i=1}^{N} (\lambda_i)^{-1} |v_i\rangle\langle v_i|$$

Check: $\left(\sum_{i=1}^{N} \lambda_i |v_i\rangle\langle v_i|\right)\left(\sum_{i=1}^{N} (\lambda_i)^{-1} |v_i\rangle\langle v_i|\right)$

$$= \sum_{i=1}^{N} \cancel{\lambda_i (\lambda_i)^{-1}} |v_i\rangle\langle v_i| = I$$

$$U^\dagger = \sum_{i=1}^{N} (\lambda_i)^* |v_i\rangle\langle v_i|$$

If $U^\dagger = U^{-1}$ then $(\lambda_i)^{-1} = \lambda_i^*$

Complex $\lambda_i = m e^{i\theta}$; $m \geq 0$ real.

$\lambda_i^* = m \cdot e^{-i\theta}$

$(\lambda_i)^{-1} = m^{-1} e^{-i\theta} \Rightarrow m = m^{-1}$ and $m = 1$.

$\Rightarrow \lambda_i = e^{i\theta}$

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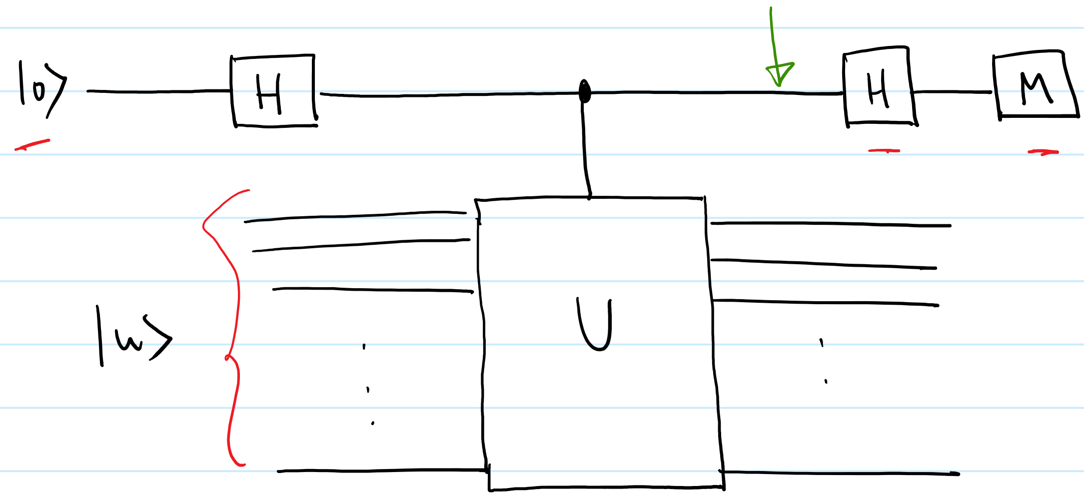
Let's just start with C-U a controlled U operation.

Consider the following circuit on input $|0\rangle|u\rangle$.



$$|0\rangle|u\rangle \xrightarrow{H} |+\rangle|u\rangle \xrightarrow{C\text{-}U} \frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle + |1\rangle\lambda|u\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle + |1\rangle|u\rangle\right)$$

$$U|u\rangle = \lambda|u\rangle$$

$$\lambda = e^{2\pi i\varphi}$$

$$|0\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$\lambda\,|1\rangle \to \lambda\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + \lambda|1\rangle\right)|u\rangle$$

$$\xrightarrow{H} \left[\frac{1+\lambda}{2}|0\rangle + \frac{1-\lambda}{2}|1\rangle\right]|u\rangle$$

After measurement:

Prob 0: $\left|\frac{1+\lambda}{2}\right|^2$ $\qquad \lambda = e^{2\pi i\varphi}$

Prob 1: $\left|\frac{1-\lambda}{2}\right|^2$

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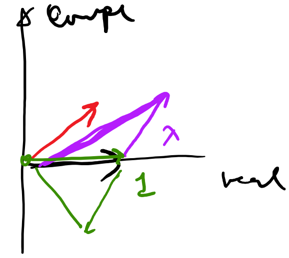

After measurement: Prob 0: $\left|\frac{1+\lambda}{2}\right|^2$ $\lambda = e^{2\pi i\varphi}$

Prob 1: $\left|\frac{1-\lambda}{2}\right|^2$

$$\frac{(1+e^{2\pi i\varphi})(1+e^{-2\pi i\varphi})}{4} = \frac{2+e^{2\pi i\varphi}+e^{-2\pi i\varphi}}{4}$$

$$= \frac{2+\cos(2\pi\varphi)+i\sin(\cancel{2\pi\varphi})+\cos(\cancel{-}2\pi\varphi)+i\sin(\cancel{-2\pi\varphi})}{4}$$

$$= \frac{1+\cos 2\pi\varphi}{2}$$ probability of 0 $\frac{m_0}{M}$

and $\frac{1-\cos 2\pi\varphi}{2}$ probability of 1. $\frac{M-m_0}{M}$

Bias of the coin is $\cos(2\pi\varphi)/2$

Estimating the bias of a coin to within ϵ with probability δ requires $O(\log(1/\delta)/\epsilon)$ Samples.

Prob Head $= 1/2 + b$.

Prob Tail $= 1/2 - b$.

Estimating $\cos(2\pi\varphi)/2$ to within m bits of precision requires $2^{\Omega(m)}$ trials.

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The exact statement for the error (which we give without proof) is:

In order to successfully obtain φ accurate to within t bits with probability $1-\epsilon$ requires:

$$\exp\left(t + \left\lceil \log\left(2 + \frac{1}{2\epsilon}\right) \right\rceil \right)$$ trials.

$.\,b_0 b_1 \ldots b_m = \frac{j}{2^m}$ phase $\lambda = e^{2\pi i \varphi}$

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Here is a version that still has complexity 2^m but doesn't require all the repeated measurements.

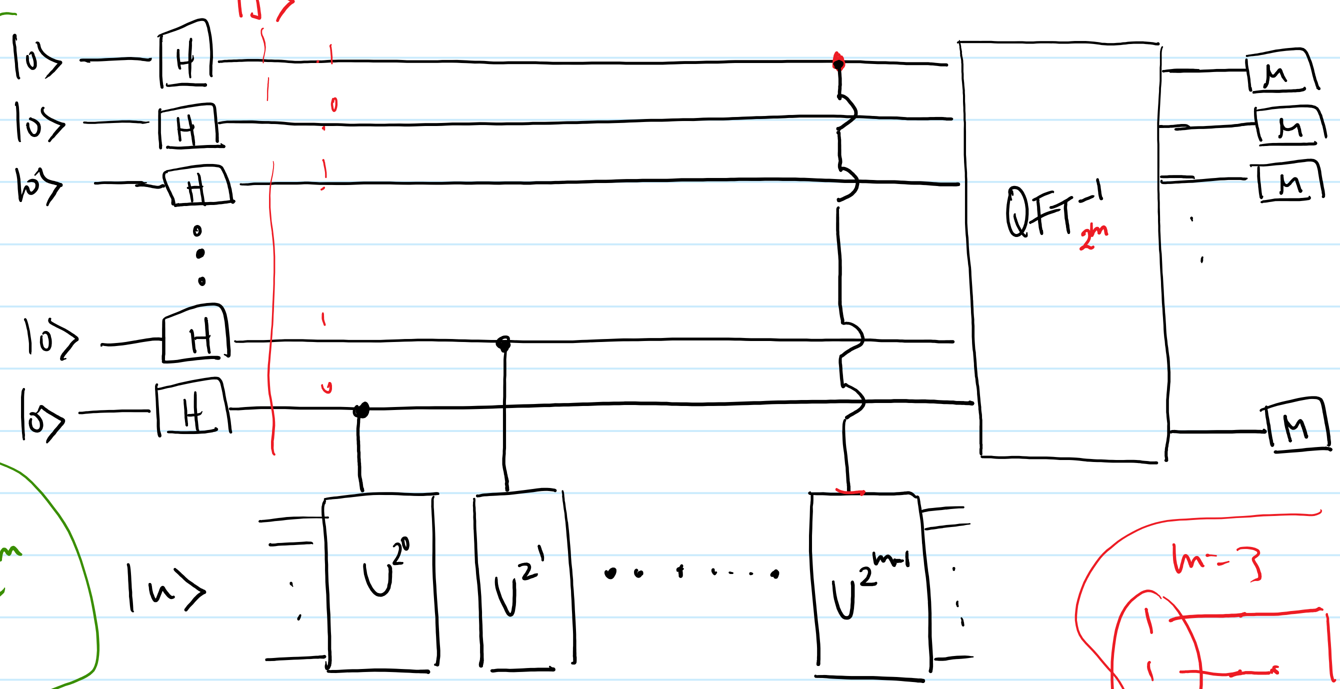
Estimating φ to within m bits of accuracy is the same as finding the integer j such that $j/2^m$ is the best approximation of φ.

$\omega = e^{2\pi i/2^m}$

→ m is a parameter we control to determine accuracy.

m qubits

poly$(n) \cdot 2^m$



$m=3$

1
1
0

$\cancel{U^{2^0}} \cdot U^{2^1} \cdot U^{2^2}$

$= U^2 \cdot U^4 \cdot U^6$

$$|0^m\rangle|u\rangle \xrightarrow{H^{\otimes m}} \left(\frac{1}{\sqrt{2^m}}\sum_{k=0}^{M-1}|k\rangle\right)\otimes|u\rangle$$

$$\xrightarrow{C-U} \frac{1}{\sqrt{M}}\sum_{k=0}^{2^m-1}|k\rangle\otimes U^k|u\rangle$$

$$\xrightarrow{C-U} \left(\frac{1}{\sqrt{2^m}}\sum_{k=0}^{M-1}\lambda^k|k\rangle\right)\otimes|u\rangle$$

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$$\left(\frac{1}{\sqrt{M}} \sum_{k=0}^{2^m-1} \lambda^k |k\rangle\right) \otimes |u\rangle$$

Now suppose $\lambda = e^{2\pi i (j/2^m)}$ — φ

Actually $\lambda = e^{2\pi i \varphi}$ and $j/2^m$ is the best approximation to φ to within $\pm \frac{1}{2^{m+1}}$

$$\left(\frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} \left(e^{2\pi i/2^m}\right)^{jk} |k\rangle\right) \otimes |u\rangle = \left[\frac{1}{\sqrt{2^m}} \sum_{k=0}^{2^m-1} \omega^{jk} |k\rangle\right] \otimes |u\rangle$$

$\omega = e^{2\pi i/2^m}$

Note that this is $FT_m |j\rangle$

So if we apply FT_m^{-1} we get:

$$\xrightarrow{(FT_m)^{-1}} |j\rangle |u\rangle$$

If we have $\varphi \approx j/2^m$, the output is j with high prob. (at least constant)

$$|\varphi - j/2^m| \leq 1/2^{m+1}$$