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A linear operator on elements of $\mathbb{C}^N$ can be expressed as an $N \times N$ matrix:

$$A|\phi\rangle = |\phi'\rangle \qquad |\phi\rangle = \begin{pmatrix} 1 \\ 2 \\ i \\ 1 \end{pmatrix}$$

$$A = \begin{bmatrix} 0 & i & 0 & 1/2 \\ i & 0 & 0 & 0 \\ 0 & 0 & 2 & 3i \\ 1/2 & 0 & 4+5i & -7i \end{bmatrix} \qquad A|\phi\rangle = \begin{pmatrix} 2i + 1/2 \\ i \\ 5i \\ \end{pmatrix}$$

The adjoint of matrix A is denoted an A^+

transpose - conjugate

$$A^+ = \begin{bmatrix} 0 & -i & 0 & 1/2 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 2 & \\ \frac{1}{2} & 0 & -3i & \end{bmatrix}$$

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$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} i & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} i \end{bmatrix}$$

Dual of $A|v\rangle$

$= \langle v| A^{\dagger}$

Check: $|v\rangle = \begin{pmatrix} 1/\sqrt{2}\, i \\ +1/\sqrt{2} \end{pmatrix}$ $\qquad (A|v\rangle) = \begin{pmatrix} -1 \\ i \end{pmatrix}$

Dual

$\begin{pmatrix} -1 & -i \end{pmatrix}$

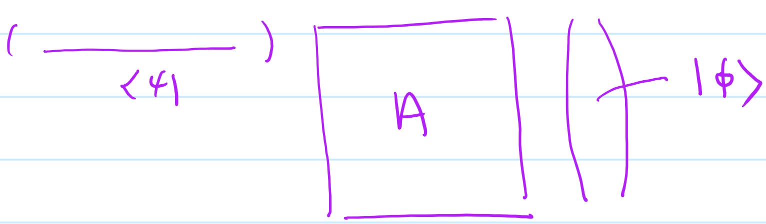
$$\underbrace{\begin{pmatrix} -1/\sqrt{2}\, i & +\frac{1}{\sqrt{2}} \end{pmatrix}}_{\langle v|} \frac{1}{\sqrt{2}} \begin{bmatrix} -i & +1 \\ -1 & -i \end{bmatrix} = \begin{pmatrix} -1 & -i \end{pmatrix}$$

Special matrix $I \in \mathbb{C}^N$ $\qquad I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$I|v\rangle = |v\rangle$

for all $|v\rangle$ $\qquad I$ = identity matrix.

Notation : $\langle \psi | A | \phi \rangle =$



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Evolution of a quantum system over time must preserve the norm of the quantum state.

U = operation performed.

for every $|\phi\rangle$ $\quad \| |\phi\rangle \| = 1 \implies \| U|\phi\rangle \| = 1$

$\langle\phi|\phi\rangle = 1 \implies \langle\phi| U^\dagger \underbrace{U|\phi\rangle} = 1$.

If $\langle\phi\, U^\dagger U\, \phi\rangle = 1$ for every $|\phi\rangle$ then $U^\dagger U = I$

Following Conditions are Equivalent:

* U is norm-preserving
* U preserves inner product
  $\langle\psi|\psi'\rangle = \langle\psi|U^\dagger U|\psi'\rangle$
* $U^\dagger U = U U^\dagger = I$
* Rows of U form an orthonormal basis
* Columns of U form an orthonormal basis

$\implies$ U is a <u>unitary</u> operator.

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$$\langle \psi | \psi' \rangle \quad \to \quad \langle \psi | \quad | \psi' \rangle$$

Outer bracket notation: $|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}$ $\quad |\psi'\rangle = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix}$

$$\underbrace{|\psi'\rangle\langle\psi|}_{\text{ket - bra}} = \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}}_{N \times 1} \underbrace{\begin{pmatrix} \alpha_1^* & \alpha_2^* & \dots & \alpha_N^* \end{pmatrix}}_{1 \times N} = \underbrace{\begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} & & \\ & & \beta_i \alpha_j^* \\ & & \end{bmatrix} \end{array}}_{N \times N}$$

Can view $|\psi'\rangle\langle\psi|$ as an operator directly:

$|\psi'\rangle\langle\psi|$ applied to $|\phi\rangle$

$$= |\psi'\rangle \underbrace{\langle\psi|\phi\rangle} \qquad \langle\psi|\phi\rangle\, |\psi'\rangle$$

Can create more complex linear operators by taking linear combinations.

$$\alpha_1 |\psi_1\rangle\langle\phi_1| + \alpha_2 |\psi_2\rangle\langle\phi_2| + \alpha_3 |\psi_3\rangle\langle\psi_3| + \dots$$

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$$|\psi\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \qquad |\phi\rangle = \begin{pmatrix} i/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix}$$

$$|\phi\rangle\langle\psi| = \begin{pmatrix} i/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} i & -i \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$$

How does this operator act on : $\begin{pmatrix} i/2 \\ \sqrt{3}/2 \end{pmatrix}$?

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix}$

$$|\phi\rangle\langle\psi| \begin{pmatrix} i/2 \\ \sqrt{3}/2 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\frac{i}{2} + \frac{\sqrt{3}}{2}\right) \begin{pmatrix} i/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{pmatrix}$$

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Orthonormal basis $|\phi_1\rangle \dots |\phi_N\rangle$ of $\mathbb{C}^N$

Any state $|v\rangle = \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle + \dots + \alpha_N|\phi_N\rangle$

What is $\langle\phi_j|v\rangle$? α_j

$$\left[\sum_{j=1}^{N} |\phi_j\rangle\langle\phi_j|\right]|v\rangle = \sum_{j=1}^{N} |\phi_j\rangle\langle\phi_j|v\rangle$$

$$(x_1 + x_2 \dots x_n)\cdot a = x_1 a + x_2 a \dots x_n a.$$

$$= \sum_{j=1}^{N} \alpha_j|\phi_j\rangle = |v\rangle$$

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Given any operator A acting on vectors in $\mathbb{C}^N$.

and any orthonormal basis $|\phi_1\rangle \ldots |\phi_N\rangle$ of $\mathbb{C}^N$

Can express A in outer-bracket notation ?
using the phi's.

$$A = \sum_{jk} \alpha_{jk} |\phi_k\rangle\langle\phi_j|$$

$$A = IAI = \left[\sum_{j=1}^{N} |\phi_j\rangle\langle\phi_j|\right] A \left[\sum_{k=1}^{N} |\phi_k\rangle\langle\phi_k|\right]$$

$$\sum_{jk}$$

$$= \sum_{j=1}^{N}\sum_{k=1}^{N} |\phi_j\rangle\underbrace{\langle\phi_j|A|\phi_k\rangle}\langle\phi_k|$$

$$\left[(x_1 + \cdots + x_N)\, a\, (y_1 + \cdots y_N)\right] = \sum_{jk} x_j a y_k.$$

$$\sum_{jk} \langle\phi_j|A|\phi_k\rangle\, |\phi_j\rangle\langle\phi_k|$$