LinAlg - Hilbert Spaces - 1

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Complex Vector space $\mathbb{C}^N$ ← N - dimension of the space.

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} \in \mathbb{C}^N$$

elements are length N vectors w/ complex entries.

Vector addition:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix} = \begin{pmatrix} \alpha_1+\beta_1 \\ \alpha_2+\beta_2 \\ \vdots \\ \alpha_N+\beta_N \end{pmatrix}$$

Scalar Multiplication:

$$c\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} = \begin{pmatrix} c\alpha_1 \\ c\alpha_2 \\ \vdots \\ c\alpha_N \end{pmatrix}$$

Linear Combination:

$$c\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix} + d\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix} = \begin{pmatrix} c\alpha_1+d\beta_1 \\ c\alpha_2+d\beta_2 \\ \vdots \\ c\alpha_N+d\beta_N \end{pmatrix}$$

$|\psi\rangle \in \mathbb{C}^N$ is a short hand for a generic column vector : $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{pmatrix}$

$$c|\psi\rangle = \begin{pmatrix} c\alpha_1 \\ \vdots \\ \alpha c_N \end{pmatrix}$$

$|\psi\rangle \quad |\phi\rangle$

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$$|0\rangle \to \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle \to \begin{pmatrix}0\\1\end{pmatrix}$$

$$|zero\rangle = \begin{pmatrix}0\\0\\ \vdots \\ 0\end{pmatrix}$$

(Not to be confused with $|0\rangle$
$0 \cdot |0\rangle + 0 \cdot |1\rangle = |zero\rangle$)

Definition The span of $\{|\psi_1\rangle, \ldots, |\psi_N\rangle\}$ is the set of all linear combinations of $|\psi_1\rangle \cdots |\psi_N\rangle$

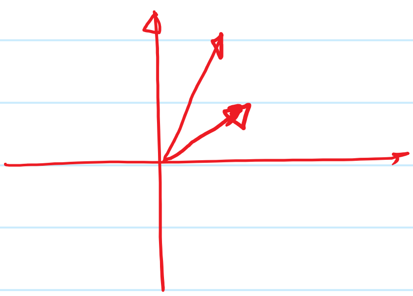
$$c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots + c_N|\psi_N\rangle \qquad c_1, c_2, \ldots, c_N \in \mathbb{C}$$

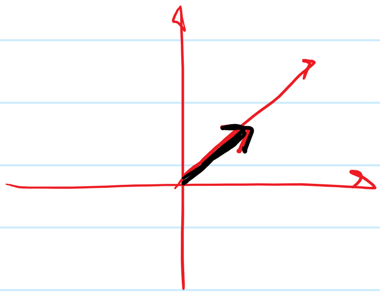
Definition: Set $\{|\psi_1\rangle, \ldots, |\psi_N\rangle\}$ is linearly independent if the only solution to:

$$c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots \; c_N|\psi_N\rangle = |zero\rangle$$

is $c_1 = c_2 = \cdots = c_N = 0$.

If a set is linearly dependent, there is a vector that can be removed without changing the span.





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A **basis** for $\mathbb{C}^N$ is a set of linearly independent vectors whose span is $\mathbb{C}^N$.

Examples of bases for $\mathbb{C}^3$

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \begin{pmatrix}2\\1\\3\end{pmatrix} \begin{pmatrix}4i\\5\\6+7i\end{pmatrix} \begin{pmatrix}-7\\0\\8i-3\end{pmatrix}$$

If $|\psi_1\rangle \dots |\psi_N\rangle$ is a basis for $\mathbb{C}^N$ then any $|\phi\rangle \in \mathbb{C}^N$ has a unique representation as:

$$c_1|\psi_1\rangle + c_2|\psi_2\rangle + \dots c_N|\psi_N\rangle$$

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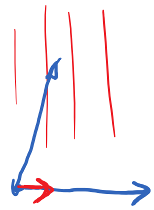

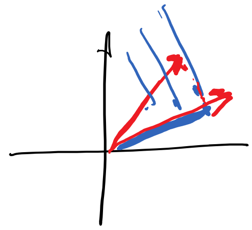

$\mathbb{C}^N$

Inner Product:

$$|\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_N \end{pmatrix} \qquad |\phi\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}$$

Inner product of $|\psi\rangle$ and $|\phi\rangle$ $= \sum_{j=1}^{N} \alpha_j^* \beta_j$

$$\alpha_1^* \beta_1 + \alpha_2^* \beta_2 + \cdots + \alpha_N^* \beta_N = \underbrace{(\alpha_1^*, \alpha_2^*, \ldots, \alpha_N^*)}_{1 \times N} \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix}}_{N \times 1} \Rightarrow 1 \times 1$$

(scalar)

Dirac notation: $\langle\psi|\phi\rangle$ = inner product of $|\psi\rangle$ and $|\phi\rangle$

Dual of $|\psi\rangle$ is $\langle\psi| = (\alpha_1^* \alpha_2^* \cdots \alpha_N^*)$

Question: What is $\langle\psi|\psi\rangle$? $\quad |\psi\rangle = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}$

$$\langle\psi| \qquad (\alpha_1^* \; \alpha_2^* \; \cdots \; \alpha_N^*) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} = \alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 + \cdots \alpha_N^*\alpha_N$$

What's the relationship between $\langle\psi|\phi\rangle$ and $\langle\phi|\psi\rangle$?

$$(\alpha_1^* \cdots - \alpha_N^*) \begin{pmatrix} \beta_1 \\ \vdots \end{pmatrix} \qquad (\beta_1^* - \cdots - \beta_N^*) \begin{pmatrix} \alpha_1 \\ \vdots \end{pmatrix}$$

$$(\alpha_1^* \beta_1)^* \to \beta_1^* \alpha_1$$

$$(\alpha_1^* \cdots \alpha_N^*) \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} \qquad (\beta_1^* \cdots \beta_N^*) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix}$$

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Tuesday, October 2, 2018 9:30 AM

$\langle \phi |$

$(1-i)(1+i) = 2$

$$|\psi\rangle = \begin{pmatrix} i/\sqrt{3} \\ 1/3 \\ 0 \\ \frac{\sqrt{2}-i}{3} \end{pmatrix} \qquad |\phi\rangle = \begin{pmatrix} 1/2 \\ \frac{1-i}{2\sqrt{2}} \\ i/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\langle \psi | = \left(-i/\sqrt{3},\ 1/3,\ 0,\ \frac{\sqrt{2}+i}{3} \right)$$

$$\langle \psi | \phi \rangle = \frac{-i}{\sqrt{3}\cdot 2} + \frac{1}{3}\cdot\left(\frac{1-i}{2\sqrt{2}}\right)$$

$$\langle \phi | \psi \rangle = \left(\quad \right)^*$$

$$\langle \phi | \phi \rangle = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1.$$

L_2 norm of $|\phi\rangle = \sqrt{\langle \phi | \phi \rangle}$ Denoted $\| |\phi\rangle \|$

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Tuesday, October 2, 2018 10:42 AM

Suppose $|\psi_1\rangle \; |\psi_2\rangle \in \mathbb{C}^N$

Define $|\phi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle \qquad c_1\, c_2 \in \mathbb{C}.$

Express $\langle\phi|$ in terms of $c_1 \;\; c_2 \;\; \langle\psi_1| \;\; \langle\psi_2|$

Suppose $|\psi_1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ $|\psi_2\rangle = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ then $|\phi\rangle = \begin{pmatrix} c_1 a_1 + c_2 a_2 \\ c_1 b_1 + c_2 b_2 \end{pmatrix}$

$$\langle\phi| = \left((c_1 a_1 + c_2 a_2)^*,\ (c_1 b_1 + c_2 b_2)^* \right)$$

$$= \left(c_1^* a_1^* + c_2^* a_2^*,\ c_1^* b_1^* + c_2^* b_2^* \right)$$

$$= c_1^* (a_1^*, b_1^*) + c_2^* (a_2^*, b_2^*)$$

$$= \langle\psi_1| c_1^* + \langle\psi_2| c_2^*$$

Dual of $|\phi\rangle = \langle\phi| = \langle\psi_1| c_1^* + \langle\psi_2| c_2^*$

$$|\phi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$$

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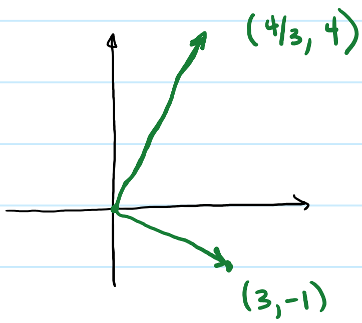
Definition

A complex vector space + inner product operation is called a Hilbert Space.

Two vectors $|\psi\rangle$ and $|\phi\rangle$ are orthogonal if $\langle\psi|\phi\rangle = 0$.



A basis $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$ of $\mathbb{C}^N$ is ortho-normal

if $\langle\psi_i|\psi_i\rangle = 1$ for all i.

$\langle\psi_i|\psi_j\rangle = 0$ if $i \neq j$.

Show that $|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $|-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

is an ortho-normal basis of $\mathbb{C}^2$.

$\langle+|+\rangle = \langle-|-\rangle = 1$

$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{2} + \frac{1}{2} = 1.$

$\langle+|-\rangle = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \frac{1}{2} - \frac{1}{2} = 0.$

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Standard basis for $\mathbb{C}^N$: $\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix} \cdots \begin{pmatrix}0\\0\\\vdots\\0\\1\end{pmatrix}$

is orthonormal.

Example 1-qubit system lives in a 2-dim Hilbert space $\mathbb{C}^2$.

Basis: $|0\rangle, |1\rangle$

$\alpha|0\rangle + \beta|1\rangle = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Check that $|0\rangle$ + $|1\rangle$ form an orthonormal basis:

$\langle 0|0\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1$

$\langle 1|1\rangle =$

$\langle 1|0\rangle = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 0$

$\langle 1|0\rangle =$

Any state can be expressed as:
$\alpha|0\rangle + \beta|1\rangle$
$= \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$

Another ortho-normal basis:

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$ $\quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$

Have already shown: $\langle +|+\rangle = \langle -|-\rangle = 1$
$\langle +|-\rangle = \langle -|+\rangle = 0.$

Can express any state $\alpha|0\rangle + \beta|1\rangle$ as $\alpha'|+\rangle + \beta'|-\rangle$ for some α' + β'.

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Example: n-qubit system lives in a 2^n-dimensional Hilbert space $\mathbb{C}^{2^n}$.

Standard Basis: $|\underline{0\cdots0}\rangle$, $|\underline{0\cdots01}\rangle$, ..., $|\underline{1\cdots1}\rangle$

each n-bit string is a state in the standard basis.

In vector form $|\underbrace{0\cdots0}_{\text{length } n}\rangle = \left.\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right\}$ length 2^n.

$$|0\cdots01\rangle = \begin{pmatrix}0\\1\\0\\\vdots\\0\end{pmatrix}$$

$$|1\cdots1\rangle = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

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Example: 3 qubits.

$$\alpha_{000}|000\rangle + \alpha_{001}|001\rangle + \alpha_{010}|010\rangle + \alpha_{011}|011\rangle + \alpha_{100}|100\rangle + \alpha_{101}|101\rangle + \alpha_{110}|110\rangle + \alpha_{111}|111\rangle$$

$$= \alpha_{000}\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} + \alpha_{001}\begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} + \alpha_{010}\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix} + \cdots + \alpha_{111}\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}\alpha_{000}\\\alpha_{001}\\\alpha_{010}\\\alpha_{011}\\\alpha_{100}\\\alpha_{101}\\\alpha_{110}\\\alpha_{111}\end{pmatrix}$$

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Monday, October 1, 2018 8:31 AM

Back to 1 qubit:

Can express any state $\alpha|0\rangle + \beta|1\rangle$ =
Uniquely as $\alpha'|+\rangle + \beta'|-\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

What are α' and β' as a function of α + β?

$$\langle +| \left(\alpha'|+\rangle + \beta'|-\rangle\right) = \alpha'\langle +|+\rangle + \beta'\cancel{\langle +|-\rangle}^{0}$$

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$\langle +| = \langle 0|\frac{1}{\sqrt{2}} + \langle 1|\frac{1}{\sqrt{2}}$

$$= \alpha'$$

$$\langle +| \left(\alpha|0\rangle + \beta|1\rangle\right) = \left(\langle 0|\tfrac{1}{\sqrt{2}} + \langle 1|\tfrac{1}{\sqrt{2}}\right)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$\frac{\alpha}{\sqrt{2}}\overset{1}{\langle 0|0\rangle} + \frac{\alpha}{\sqrt{2}}\cancel{\langle 1|0\rangle} + \frac{\beta}{\sqrt{2}}\cancel{\langle 0|1\rangle} + \frac{\beta}{\sqrt{2}}\overset{1}{\langle 1|1\rangle}$$

$$= \frac{\alpha + \beta}{\sqrt{2}} = \alpha'$$

$\alpha|0\rangle + \beta|1\rangle$

$= \frac{\alpha+\beta}{\sqrt{2}}|+\rangle + \frac{\alpha-\beta}{\sqrt{2}}|-\rangle$

What is β' in terms of α and β?

$$\langle -| \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$\left(\langle 0|\tfrac{1}{\sqrt{2}} + \langle 1|\tfrac{-1}{\sqrt{2}}\right)\left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$\frac{\alpha-\beta}{\sqrt{2}} = \beta'$$

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In general if $|\phi_1\rangle \dots |\phi_N\rangle$ is an orthonormal basis of $\mathbb{C}^N$

State $\alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle + \dots + \alpha_N|\phi_N\rangle$

If $|\psi_1\rangle, \dots, |\psi_N\rangle$ is also an orthonormal basis of $\mathbb{C}^N$.

Can express this same state as $\underline{\beta_1|\psi_1\rangle + \beta_2|\psi_2\rangle + \dots + \beta_N|\psi_N\rangle}$

What is β_j?

$$\langle\psi_j| \quad \beta_1|\psi_1\rangle + \beta_2|\psi_2\rangle + \dots + \beta_N|\psi_N\rangle$$

$$\beta_j = \langle\psi_j|\ \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle + \dots + \alpha_N|\phi_N\rangle$$

$$\alpha_1\langle\psi_j|\phi_1\rangle + \alpha_2\langle\psi_j|\phi_2\rangle + \dots + \alpha_N\langle\psi_j|\phi_N\rangle$$

$N \times N$ matrix A

$A^\dagger$ = transpose, conjugate.

$$A = \begin{bmatrix} 7 & 4+i & 8i \\ 3 & 6-2i & 2 \\ 5 & 4 & 1+2i \end{bmatrix} \qquad A^\dagger = \begin{matrix} 7 & 3 & 5 \\ 4-i & 6+2i & 4 \\ -8i & 2 & 1-2i \end{matrix}$$

If A is an $N \times N$ matrix then the columns of A form an orthonormal basis of $\mathbb{C}^N$ if and only if:

identity matrix $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

$$A^\dagger \cdot A = I$$

$$\underset{A^\dagger}{\begin{bmatrix} \langle \phi_1 | \\ \langle \phi_2 | \\ \vdots \\ \langle \phi_N | \end{bmatrix}} \underset{A}{\begin{bmatrix} |\phi_1\rangle & |\phi_2\rangle & \cdots & |\phi_N\rangle \end{bmatrix}} = \begin{array}{c c} & j \\ i & \begin{bmatrix} & \vdots \\ \cdots & \langle \phi_i | \phi_j \rangle \end{bmatrix} \end{array}$$