Error Correction (Shor) - page 1

$\alpha|0\rangle + \beta|1\rangle$

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Classical error correction is accomplished by encoding data in a redundant way so that if part of the data is lost, the problem can be detected and then corrected.

Challenging in the quantum setting because:

1) Data can not be copied (No Cloning)
2) Measurement destroys quantum information.
3) Errors can be continuous.

For a single qubit, one possible error is a bit flip:

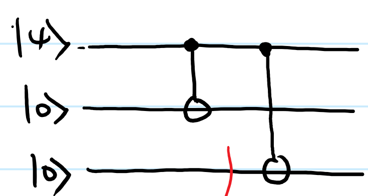
$$|0\rangle \to |1\rangle \quad , \quad |1\rangle \to |0\rangle .$$

More generally: $\alpha|0\rangle + \beta|1\rangle \to \beta|0\rangle + \alpha|1\rangle$

We will use a standard repitition code:

$|0\rangle \to |000\rangle$

$|1\rangle \to |111\rangle$



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi\rangle|00\rangle \to$

$\alpha|000\rangle + \beta|111\rangle$

$(\alpha|00\rangle + \beta|11\rangle) \otimes |0\rangle$

$\alpha|000\rangle + \beta|111\rangle$

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$I = \sum_x |x\rangle\langle x|$

$-P_1$

We can use the following projectors to detect an error:

$P_0 = |000\rangle\langle 000| + |111\rangle\langle 111|$

$P_1 = |100\rangle\langle 100| + |011\rangle\langle 011|$

$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|$

$P_3 = |001\rangle\langle 001| + |110\rangle\langle 110|$

$P_1 |000\rangle = 0$

$P_2 |010\rangle = |010\rangle$

$P_3 (\alpha|001\rangle + \beta|110\rangle)$

We can store the result in an auxiliary register:

$\underbrace{|\text{———}\rangle}_{\text{data}} \underbrace{|000\rangle}_{\text{syndrome}}$

$\alpha|000\rangle + \beta|111\rangle$

Can do $P_1 \otimes XII + (I - P_1) \otimes I^3$

$(\alpha|100\rangle + \beta|011\rangle)|000\rangle \longrightarrow (\alpha|100\rangle + \beta|011\rangle)|100\rangle$

Note that measuring the syndrome does not destroy the quantum information in the first register. It just indicates that an error has occurred.

Then we can do a controlled X to correct the error: 1st bit of syndrome is control
1st bit of data is the target

This works even if error happens in superposition.

Flip bit 1 w/ amplitude $1/\sqrt{2}$

Flip bit 2 w/ amplitude $1/\sqrt{2}$

$\frac{1}{\sqrt{2}}(XII + IXI)$

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$$\frac{1}{\sqrt{2}}\left(\alpha|100\rangle + \beta|011\rangle\right)|100\rangle + \frac{1}{\sqrt{2}}\left(\alpha|010\rangle + \beta|101\rangle\right)|010\rangle$$

After error correction:

$$\left(\alpha|000\rangle + \beta|111\rangle\right)\frac{1}{\sqrt{2}}\left(|100\rangle + |010\rangle\right)$$

Can measure syndrome before or after.

This method does not work with 2 or more errors
$\Rightarrow$ Only 0 or 1 errors.

If each qubit is flipped independently with probability p, the probability that this method works is:

$$\underbrace{(1-p)^3}_{\text{No errors}} + \underbrace{3p(1-p)^2}_{\text{1 error.}} = 1 - 3p^2 + 2p^3$$

(prob error in 1st qubit only is $p(1-p)^2$).

This method is an improvement over no error correction if:

$$3p^2 - 2p^3 > p \Rightarrow \quad p < 1/2$$

An alternative view:

Instead of performing measurements P_0 P_1 P_2 P_3

Do two measurements: Z_1Z_2I and IZ_2Z_3

$$Z_1Z_2I = \left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)\otimes I - \left(|01\rangle\langle 01| + |10\rangle\langle 10|\right)\otimes I$$

This has eigenvalues ± 1 and gives 1 bit of information.

Z_1Z_2I and IZ_2Z_3 together give 2 bits of information.

Note Z_1Z_2 compares qubits 1 and 2.
eigenvalue is $+1$ if qubits 1 & 2 are equal.
eigenvalue is -1 if qubits 1 & 2 are different

$$\begin{bmatrix} 1 &&&&&&& \\ & -1 &&&&& 0 & \\ && -1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ & 0 &&&& -1 && \\ &&&&&& -1 & \\ &&&&&&& 1 \end{bmatrix}$$

The two bits of information tell you if & where a single bit flip occurred:

Z_1Z_2I	IZ_2Z_3	
+1	+1	no flip.
+1	-1	3rd
-1	+1	1st
-1	-1	2nd

No information about α or β is obtained.

The syndrome is the two bits resulting from this measurement.

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Another possible error is a phase flip:

with probability p, Z operator is applied to a qubit.

$$\alpha|0\rangle + \beta|1\rangle \longrightarrow \alpha|0\rangle - \beta|1\rangle.$$

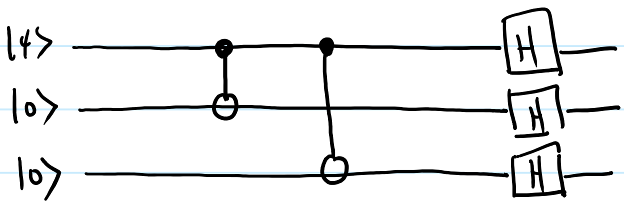
Can turn a phase flip channel into a bit flip channel

Z flips $|+\rangle$ and $|-\rangle$

Encode $|0\rangle$ as $|+++\rangle$

Encode $|1\rangle$ as $|---\rangle$

Analysis is the same as with the bit flip channel. Measurements done in the $|+\rangle$ $|-\rangle$ basis.



Syndrome measurement is done by

$$H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = X_1 X_2$$
$$H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$$

Recovery on the first qubit is done by $H_1 X_1 H_1 = Z_1$

For example, if syndrome indicated that $|+++\rangle$ became $|-++\rangle$ then $Z_1|-++\rangle = |+++\rangle$

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The Shor Code uses 9 qubits to encode a single qubit and protect against any error as long as it effects only a single qubit of the code.

First encode the qubit using the phase-flip code then encode each of the qubits using the bit-flip code.

This method of encoding using nested levels is called concatenation and is useful for combining benefits of different codes.

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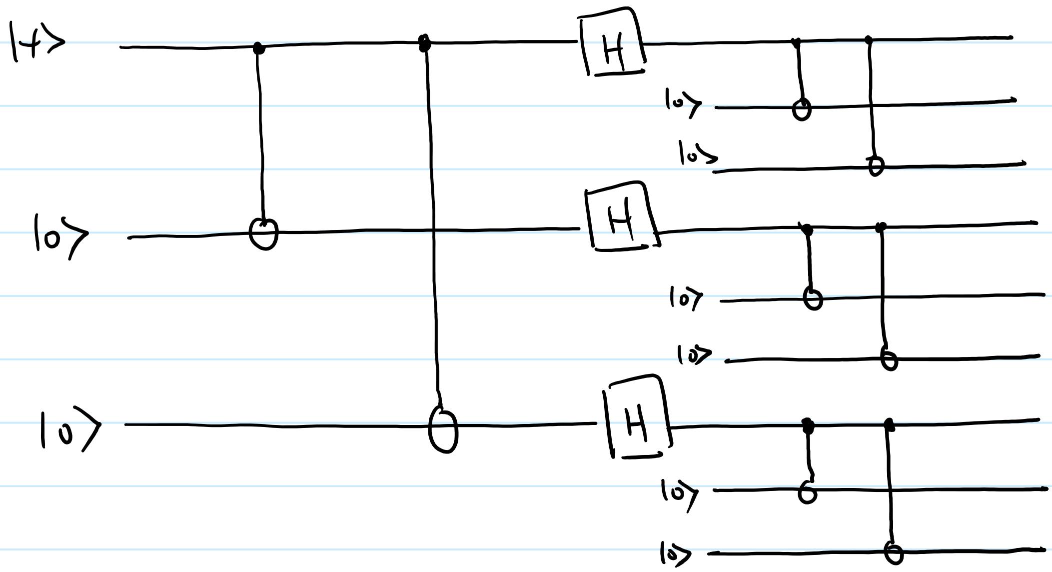
$$|0\rangle \longrightarrow \frac{(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)(|000\rangle+|111\rangle)}{2\sqrt{2}}.$$

$$|1\rangle \longrightarrow \frac{(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)(|000\rangle-|111\rangle)}{2\sqrt{2}}$$



To correct for bit flips, use the error correction procedure for bit flips in each block separately:

Z_1Z_2 & $Z_2Z_3 \longrightarrow$ correct

Z_4Z_5 & $Z_5Z_6 \longrightarrow$ correct

Z_7Z_8 & $Z_8Z_9 \longrightarrow$ correct

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$$|0\rangle \longrightarrow \frac{(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}.$$

$$|1\rangle \longrightarrow \frac{(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Now correct for phase flips between blocks

Measure: $X_1X_2X_3X_4X_5X_6$
$X_4X_5X_6X_7X_8X_9$

For example suppose there is a phase flip in the first block:

$$|0\rangle \longrightarrow \frac{(|000\rangle - |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)}{2\sqrt{2}}.$$

$$|1\rangle \longrightarrow \frac{(|000\rangle + |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)}{2\sqrt{2}}$$

Measuring $X_1X_2X_3X_4X_5X_6$ will yield -1

To correct, apply $Z_1Z_2Z_3$

$$\alpha(|000\rangle - |111\rangle) + \beta(|000\rangle + |111\rangle)$$

$$\xrightarrow[Z_1Z_2Z_3]{} \alpha(|000\rangle + |111\rangle) + \beta(|000\rangle - |111\rangle).$$

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If the error is a bit flip and phase flip on the same qubit $X_1 Z_1$

each correction procedure (bit flip then phase flip) will work independently.

Any 1-qubit error can be expressed as:

$$E = e_0 I + e_1 X_1 + e_2 Z_1 + e_3 X_1 Z_1$$

$$E|\psi\rangle = e_0|\psi\rangle + e_1 X_1|\psi\rangle + e_2 Z_1|\psi\rangle + e_3 X_1 Z_1|\psi\rangle$$

Measuring the syndrome collapses to one of the four states:

$$|\psi\rangle \quad X_1|\psi\rangle \quad Z_1|\psi\rangle \quad X_1 Z_1|\psi\rangle$$

And the appropriate recovery can be applied to get back to $|\psi\rangle$.

A whole continuum of errors can be handled by only correcting a discrete set.