Friday, October 5, 2018 10:26 AM

# Eigenvalues + Eigenvectors

linear operator $A$

$|\phi\rangle$ is an eigenvector of $A$ if

$$\overset{\mathbb{C}}{A}|\phi\rangle = \overset{\mathbb{C}}{\lambda}|\phi\rangle$$

$\mathbb{C}$ scalar $\lambda$ is called an eigenvalue of $A$.

If $A$ is linear operator acting on vectors in $\mathbb{C}^N$, then there is a set of eigenvectors that forms an orthonormal basis of $\mathbb{C}^N$.

$\{|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_N\rangle\}$ eigenvalues of $A$

also an orthonormal basis of $\mathbb{C}^N$.

$i \neq j$ $\quad \langle\phi_i|A|\phi_j\rangle = \langle\phi_i|\lambda_j|\phi_j\rangle = \lambda_j \cancelto{0}{\langle\phi_i|\phi_j\rangle}$

$$\langle\phi_i|A|\phi_i\rangle = \langle\phi_i|\lambda_i|\phi_i\rangle = \lambda_i \underline{\langle\phi_i|\phi_i\rangle} = \lambda_i.$$

When we express $A$ using basis $\{|\phi_i\rangle\}$

$$\sum_{jk} \langle\phi_j|A|\phi_k\rangle \; |\phi_j\rangle\langle\phi_k| = \sum_{j=1}^{N} \lambda_j |\phi_j\rangle\langle\phi_j|$$

Theorem: $A$ has a diagonal representation iff $A$ is normal: $AA^\dagger = A^\dagger A$.

# Eig-Meas-page 2

Friday, October 5, 2018 10:37 AM

Example: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

eigval: $1$ $-1$

Eigenvectors $|+\rangle$ and $|-\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} 1/\sqrt{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$X = |+\rangle\langle+| - |-\rangle\langle-|$$

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

# Eig-Meas-page 3

Sunday, October 7, 2018 12:18 PM

If $|v_i\rangle$ and $|v_j\rangle$ are eigenvectors of $A$ with different eigenvalues $(\lambda_i \neq \lambda_j)$, then $\langle v_i | v_j \rangle = 0$.

Non-degenerate case

Example: $A$ is a $4\times 4$ matrix

$\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$
(all distinct)

$\lambda_1 \quad |v_1\rangle$
$\lambda_2 \quad |v_2\rangle$
$\lambda_3 \quad |v_3\rangle$
$\lambda_4 \quad |v_4\rangle$

If $|v_i\rangle$'s are normalized then the eigen vectors are all unique.

and they form an orthonormal basis

Degenerate Case.

$\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4$

$\lambda_1 - |v_1\rangle$

$\lambda_2 = \lambda_3$ — $|v_2\rangle$, $|v_3\rangle$

$\lambda_4 - |v_4\rangle$

$|v_1\rangle \; |w_2\rangle \; |w_3\rangle \; |v_4\rangle$

Are an orthonormal basis of eigenvectors if:

$|w_2\rangle$ and $|w_3\rangle$ form an orthonormal basis of the space spanned by $|v_2\rangle$ + $|v_3\rangle$

$|v_2\rangle \qquad |v_3\rangle \qquad \langle v_2 | v_3 \rangle = 0$

$\frac{1}{\sqrt{2}}\left(|v_2\rangle + |v_3\rangle\right) \qquad \frac{1}{\sqrt{2}}\left(|v_2\rangle - |v_3\rangle\right).$

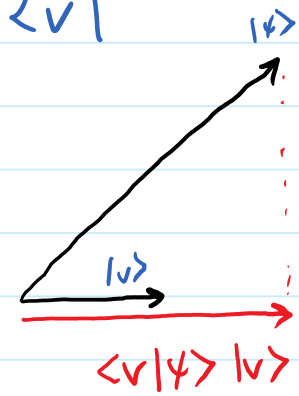
Friday, October 5, 2018 10:42 AM

**Projectors:** (Special class of linear operators in which eigenvalues $= 1$ or $0$).

Vector $|v\rangle$ (Normalized : $\langle v|v\rangle = 1$).

Projector onto $|v\rangle$ : $P_v = |v\rangle\langle v|$

$$P_v|\psi\rangle = \langle v|\psi\rangle\,|v\rangle$$




Can also project on to subspaces of higher dimension :

$|v_1\rangle \ldots |v_r\rangle \in \mathbb{C}^N$ $\qquad \langle v_j|v_k\rangle = \delta_{kj}$

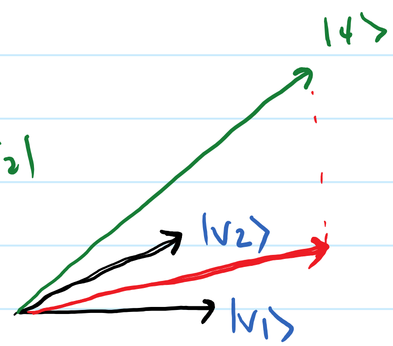
$\delta_{kj} = 1$ if $k = j$

$\delta_{kj} = 0$ if $k \neq j$.

Orthonormal set (generally does NOT span $\mathbb{C}^N$).

$$P = \sum_{j=1}^{r} |v_j\rangle\langle v_j|$$

$|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|$




Note:
$P^2 = P$.

# Eig-Meas-page 5

Friday, October 5, 2018 11:00 AM

Measurement revisited

Measurement consists of an orthonormal basis $\{|v_j\rangle\}$ and a real value $r_j$ for each $|v_j\rangle$.

(Assume for now that $r_j$'s are all distinct)

Measure state $|\phi\rangle$ in basis $\{|v_j\rangle\}$

Outcome is $r_j$ with prob $|\langle v_j|\phi\rangle|^2$

$\langle v_j|$ $|\phi\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \cdots + \alpha_N|\alpha_N\rangle$ $\langle v_j|\phi\rangle = \alpha_j$

$\alpha_1\langle v_j|v_1\rangle + \alpha_2\langle v_j|v_2\rangle \cdots \alpha_j\underline{\langle v_j|v_j\rangle}$

Afterwards state is $|v_j\rangle$.

1 qubit example: $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

Measure in $|0\rangle$ $|1\rangle$ basis.

0 1

Probability outcome is 0 : $|\langle 0|\phi\rangle|^2$

$= |\alpha|^2$

$|\alpha|^2 + |\beta|^2 = 1$.

Afterwards, state is $|0\rangle$.

# Eig-Meas-page 6

Friday, October 5, 2018 11:05 AM

Could also measure $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

in the $|+\rangle$ $|-\rangle$ basis.

Outcome is "+" with prob. $|\langle +|\phi\rangle|^2$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\underbrace{\left(\langle 1|\tfrac{1}{\sqrt{2}} + \langle 0|\tfrac{1}{\sqrt{2}}\right)}_{\langle +|}\underbrace{\left(\alpha|0\rangle + \beta|1\rangle\right)}_{|\phi\rangle}$$

$\cancel{\frac{\alpha}{\sqrt{2}}\langle 1|0\rangle}$

$$= \frac{\beta}{\sqrt{2}} + \frac{\alpha}{\sqrt{2}} = \frac{\beta+\alpha}{\sqrt{2}}.$$

prob $|+\rangle$ $\quad |\langle +|\phi\rangle|^2 = \left|\frac{\beta+\alpha}{\sqrt{2}}\right|^2 \rightsquigarrow |+\rangle$

prob $|-\rangle$ $\quad |\langle -|\phi\rangle|^2 = \left| - \left|\frac{\beta+\alpha}{\sqrt{2}}\right|^2 \right. \rightsquigarrow |-\rangle$

state after measurement.

Sunday, October 7, 2018 12:35 PM

3 qubit example:

$$\alpha_0 |000\rangle + \alpha_1 |001\rangle + \alpha_2 |010\rangle + \alpha_3 |011\rangle$$
$$+ \alpha_4 |100\rangle + \alpha_5 |101\rangle + \alpha_6 |110\rangle + \alpha_7 |111\rangle$$

Measure all 3 qubits in the standard basis:

$|v_0\rangle = |000\rangle$ $r_0 = 0$

$|v_1\rangle = |001\rangle$ $r_1 = 1$

$|v_2\rangle = |010\rangle$ $r_2 = 2$

$|v_3\rangle = |011\rangle$ $r_3 = 3$

$|v_4\rangle = |100\rangle$ $r_4 = 4$

$|v_5\rangle = |101\rangle$ $r_5 = 5$

$|v_6\rangle = |110\rangle$ $r_6 = 6$

$|v_7\rangle = |111\rangle$ $r_7 = 7.$

Outcome is $j$ w.p. $|\langle v_j | \phi\rangle|^2 = |\alpha_j|^2$

Afterwards the state is $|v_j\rangle$

# Eig-Meas-page 8

Sunday, October 7, 2018 12:44 PM

Measure just the first qubit:

degenerate case
$r_j$'s not all distinct.

$$|v_0\rangle = |000\rangle$$
$$|v_1\rangle = |001\rangle$$
$$|v_2\rangle = |010\rangle$$
$$|v_3\rangle = |011\rangle$$

$r_0 = 0$

$$P_0 = |v_0\rangle\langle v_0| + |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| + |v_3\rangle\langle v_3|$$

projector onto the subspace of all states that have "0" as first qubit.

$$|v_4\rangle = |100\rangle$$
$$|v_5\rangle = |101\rangle$$
$$|v_6\rangle = |110\rangle$$
$$|v_7\rangle = |111\rangle$$

$r_1 = 1$

$$P_1 = |v_4\rangle\langle v_4| + |v_5\rangle\langle v_5| + |v_6\rangle\langle v_6| + |v_7\rangle\langle v_7|$$

State: $|\phi\rangle$ outcome is 0 w.p. $\big|P_0|\phi\rangle\big|^2$

afterwards state is $\dfrac{P_0|\phi\rangle}{\big|P_0|\phi\rangle\big|}$

$$P_0|\phi\rangle =$$

$$\left(|v_0\rangle\langle v_0| + |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| + |v_3\rangle\langle v_3|\right)$$
$$\alpha_0|v_0\rangle + \alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \alpha_3|v_3\rangle + \alpha_4|v_4\rangle + \alpha_5|v_5\rangle + \alpha_6|v_6\rangle + \alpha_7|v_7\rangle$$

$$\alpha_0 \overbrace{\langle v_0|v_0\rangle}^{1}|v_0\rangle + \alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \alpha_3|v_3\rangle$$

$$\big|P_0|\phi\rangle\big|^2 = |\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2$$

Final state is $$\frac{\alpha_0|v_0\rangle + \alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \alpha_3|v_3\rangle}{\left(|\alpha_0|^2 + |\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2\right)^{1/2}}.$$

Sunday, October 7, 2018 12:59 PM

## Generalized Measurement (degenerate case)

Orthonormal basis $|v_1\rangle \ldots |v_N\rangle$
outcomes $r_1 \ldots r_N$ (real)

$r_j$'s not all distinct.

$|v_1\rangle \cdots |v_m\rangle$ all have $r_j = R$ $1 \le j \le m$.

$$P_R = \sum_{j=1}^{m} |v_j\rangle\langle v_j|$$

projector onto subspace spanned by $|v_1\rangle \ldots |v_m\rangle$

Outcome of measurement $= R$ with prob

$$\left| P_R |\phi\rangle \right|^2$$

Afterwards New state is $\dfrac{P_R|\phi\rangle}{|P_R|\phi\rangle|}$

$$|\phi\rangle = \alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \cdots \alpha_m|v_m\rangle + \alpha_{m+1}|v_{m+1}\rangle + \cdots + \alpha_N|v_N\rangle$$

Prob. outcome $= R$: $|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_m|^2$

State becomes $\dfrac{\alpha_1|v_1\rangle + \alpha_2|v_2\rangle + \cdots + \alpha_m|v_m\rangle}{\left(|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_m|^2\right)^{1/2}}$

Sunday, October 7, 2018 1:05 PM

# Postulate of Quantum Mechanics : Measurement

Measurement operator (combines $v_j$'s + $|v_j\rangle$'s)

$$M = \sum_j v_j \, |v_j\rangle\langle v_j|$$

$|v_j\rangle$'s are eigenvectors of M.
$v_j$'s are eigenvalues.

M is Hermetian (eigenvalues are real) $M = M^\dagger$

⇒ Every physically observable quantity:
(Energy, momentum, location, ...)
is associated with a Hermetian operator

The outcome of measuring the quantity is an eigenvalue of the operator

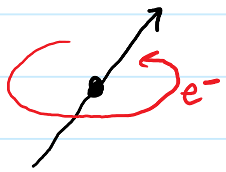
Afterwards the state $|\phi\rangle$ becomes $P|\phi\rangle$ (normalized)

P is the projector onto the subspace spanned by the eigenvectors whose eigenvalues is the same as the outcome.

Sunday, October 7, 2018 1:22 PM

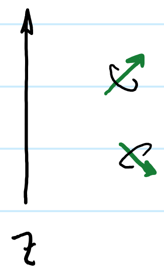
# Stern- Gerlach Experiment

1922 - Silver
1927 - Hydrogen.




Atomic spin: Loop of "current" creates a magnetic dipole moment
(bar magnet)

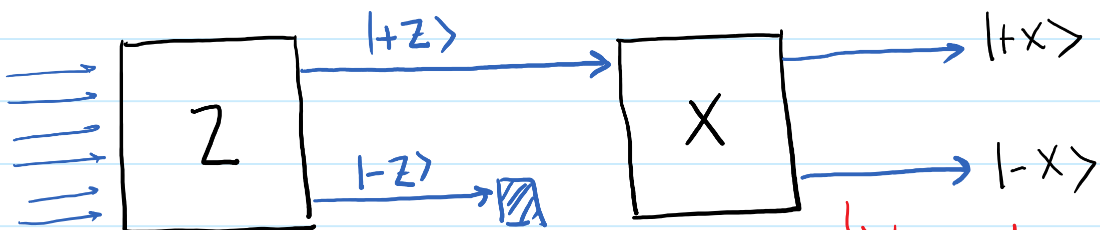
Experimental aparatus: direction of deflection depends on $\hat{z}$ component of the atom's magnetic dipole moment $\hat{z}$ component is measured.




Would expect this to be uniformly distributed

Instead: produced two discrete beams corresponding to:
↑ and ↓.

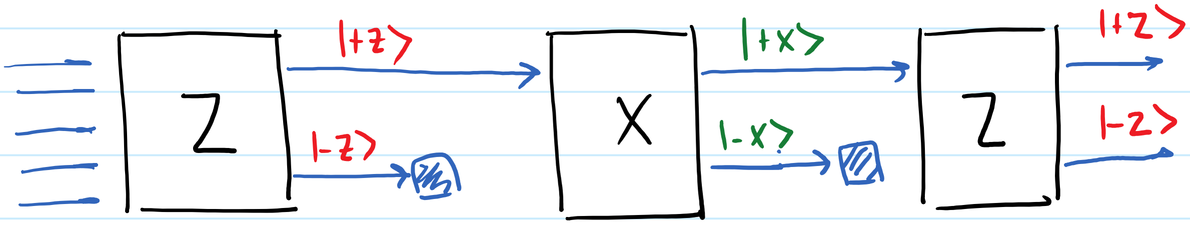
Cascade two deflectors
Second turned on its side - measures spin in the $\hat{x}$ direction.




↳ Would have expected a single beam
$|+Z\rangle$ has X-component = 0.

Sunday, October 7, 2018 1:22 PM

Perhaps atoms have magnetic dipole moment along each axis independently




Quantum Explanation: $|+z\rangle = |0\rangle$ $|-z\rangle = |1\rangle$

Z measures spin in $|0\rangle$ $|1\rangle$ basis

X measures spin in $|+\rangle$ $|-\rangle$ basis.