CS 190/264: Quantum Computation *Instructor: Sandy Irani*

Homework 3

Due: October 24, 2018, 2PM

Note: graduate students are not required to do problem 1, but are required to probelm 6. Undergraduates are required to do problems 1-5.

1. Consider the game played by Alice and Bob in Bell's protocol. Suppose that the random bit that Alice receives $X_A = 1$ and the random bit that Bob receives is $X_B = 0$. Give the probabilities for each of combination for Alice and Bob's output bits. That is, give the probability that $a = 0$ and $b = 0$. Then do the same for the other three possible values for a and b. What's the probability that Alice and Bob win the game?

2. Work out a version of the quantum teleportation protocol if Bob and Alice are given the entangled pair $1/\sqrt{2}(|01\rangle - |10\rangle)$ instead of $1/\sqrt{2}(|00\rangle + |11\rangle)$.

3. Normally, we consider two quantum states that differ by a multiple of a "global phase" $e^{i\theta}$ to be equivalent, ($|\phi\rangle \equiv e^{i\theta}|\phi\rangle$), because any measurement performed on $|\phi\rangle$ will have the same likelihoods and outcomes as a measurement on $e^{i\theta}|\phi\rangle$. Thus, the factor of $e^{i\theta}$ is undetectable by any measurement.

   (a) Prove that it is possible to multiply any normalized 1-qubit state by a phase $e^{i\theta}$ so that the state is in the form $a|0\rangle + e^{i\gamma}b|1\rangle$, where a and b are non-negative real numbers that satisfy $a^2 + b^2 = 1$.

   (b) Let $|v\rangle = a|0\rangle + e^{i\gamma}b|1\rangle$ be a normalized 1-qubit state. Define $|v^{\perp}\rangle$ to be the normalized state that is perpendicular to $|v\rangle$. The state $|v^{\perp}\rangle$ will be unique up to a global phase. Express $|v^{\perp}\rangle$ in the standard basis.

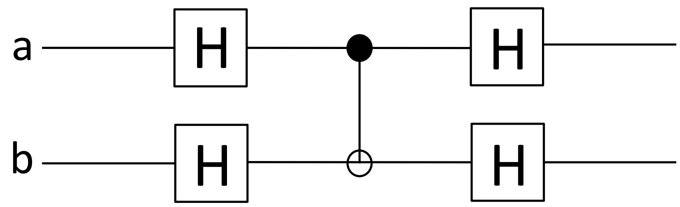
   (c) In class, we showed that the state $|\Phi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ can be expressed as $1/\sqrt{2}(|\psi\rangle|\psi\rangle + |\psi^{\perp}\rangle|\psi^{\perp}\rangle)$, for any $|\psi\rangle$ such that $|\psi\rangle$ has real amplitudes in the standard basis. That is, $|\psi\rangle = a|0\rangle + b|1\rangle$, where a and b are real. This is in general not true if the state $|\psi\rangle$ has a phase: $|\psi\rangle = a|0\rangle + e^{i\gamma}b|1\rangle$.

   Prove that the Bell state $|\Psi^{-}\rangle = 1/\sqrt{2}(|01\rangle - |10\rangle)$ can be expressed as $|\Psi^{-}\rangle = \frac{1}{\sqrt{2}}(|vv^{\perp}\rangle - |v^{\perp}v\rangle)$ for any 1-qubit state $|v\rangle$, up to a global phase. That is, it is OK to show that $e^{i\theta}|\Psi^{-}\rangle = \frac{1}{\sqrt{2}}(|vv^{\perp}\rangle - |v^{\perp}v\rangle)$, for some θ.

4. (a) Describe the action of a CNOT gate if the target bit is $|-\rangle$.

   (b) Describe the action of a CNOT gate if the target bit is $|+\rangle$.

   (c) Now show that the following circuit is effectively a CNOT gate with the control and target qubits swapped (i.e. b is the control and a is the target).



5. The single qubit gate U_θ computes a rotation between the $|0\rangle$ and $|1\rangle$ states:

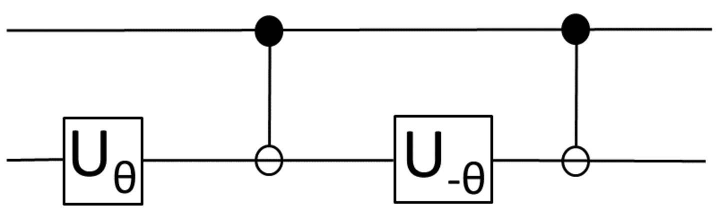
$$U_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The circuit below uses a CNOT gate as well as a U_θ gate.

(a) What is the output on the circuit when the input is $|0\rangle \otimes |\phi\rangle$, where $|\phi\rangle$ is an arbitrary 1-qubit state?

(b) What is the output on the circuit when the input is $|1\rangle \otimes |\phi\rangle$, where $|\phi\rangle$ is an arbitrary 1-qubit state? (Hint: you will need the double-angle formulas from trigonometry.)

(c) Describe in words what the circuit does for a general input state.



For graduate students: skip problem 1, and do the following problem:

6. Suppose that a 2-qubit state is shared by Alice and Bob. Suppose that Alice performs a unitary opertion U on her qubit and then Bob measures his qubit in some basis $\{|\phi\rangle, |\phi^\perp\rangle\}$. Show that the probabilities of the outcomes from Bob's measurement do not depend on the unitary operation chosen by Alice.