CS 190/264: Quantum Computation *Instructor: Sandy Irani*

Homework 2

Due: October 17, 2018, 2PM

Note: graduate students are not required to do problems 3-6, but are required to probelm 9, 10, and 11. Undergraduates are required to do problems 1-8.

1. Suppose a Hermetian operator A that operatorates on vectors in $\mathbb{C}^4$ has eigenvectors $|v_1\rangle$, $|v_2\rangle$, $|v_3\rangle$, $|v_4\rangle$. Moreover the set $|v_1\rangle, |v_2\rangle, |v_3\rangle, |v_4\rangle$ form an ortho-normal basis of $\mathbb{C}^4$. The eigenvalue of $|v_i\rangle$ is λ_i. Suppose that $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4$. Give a different set of four egenvectors of A that also form an ortho-normal basis of $\mathbb{C}^4$.

2. Consider a linear operator A with eigenstates $|v_1\rangle, \ldots, |v_N\rangle$, and corresponding eigenvalues $\lambda_1, \ldots, \lambda_N$.

   (a) Express A in outer-bracket notation in the $\{|v_i\rangle\}$ basis.

   (b) Let A^2 be the linear operator obtained by applying A twice. The matrix representation of A^2 is just the matrix representation of A squared. Express A^2 in outer-bracket notation in the $\{|v_i\rangle\}$ basis. (Hint: take your outer-bracket notation for A in the previous question and see what happens when you apply it twice. Then simplify the expression as much as possible.)

3. Suppose the state $|\phi\rangle = \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$ is measured in the $|+\rangle$, $|-\rangle$ basis. What is the probability of each outcome and what is the state afterwards (depending on the outcome of the measurement)?

4. Consider a two-qubit system. Suppose that the second qubit is measured.

   (a) Give an expression for the projector P_0 which projects onto the subspace spanned by the states in which the outcome of the measurement is 0. Do the same for P_1 which projects onto the subspace spanned by the states in which the outcome of the measurement is 1.

   (b) For the state

   $$|\phi\rangle = \frac{1+i}{3}|00\rangle + \frac{\sqrt{2}}{3}|01\rangle + \frac{-i\sqrt{2}}{3}|10\rangle + \frac{-1}{\sqrt{3}}|11\rangle,$$

   give the probability of each outcome of the measurement and the state afterwards.

5. Express each of the following linear opeartors as a 4×4 matrix.

   (a) $X \otimes I$

   (b) $I \otimes X$

   (c) $Z \otimes X$

6. Give the result of each operator applied to the state $|\Phi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.
   (a) $X \otimes I$
   (b) $I \otimes X$
   (c) $Z \otimes X$

7. Consider an n-qubit system.
   (a) Let $I_{2,3,\ldots,n}$ denote the identity operator applied to qubits 2 through n. What are the dimensions of $I_{2,3,\ldots,n}$ in matrix form?
   (b) Give a schematic representation of $H \otimes I_{2,3,\ldots,n}$.
   (c) Give a schematic representation of $I_{1,2,\ldots,n-1} \otimes H$, where $I_{1,2,\ldots,n-1}$ is the identity operator applied to qubits 1 through $n-1$.

8. If A is a linear operator, then the matrix representation of $A^\dagger$ is obtained by taking the conjugate-transpose of the matrix representation of A. To get the conjugate-transpose of a matrix, take the transpose of the matrix and then take the complex conjugate of each entry in the matrix.
   (a) What is $|v\rangle^\dagger$ (the conjugate-transpose of $|v\rangle$)?
   (b) For any two matrices A and B, if the number of columns of A is equal to the number of rows of B, then $(AB)^\dagger = B^\dagger A^\dagger$. Use this fact to show that $(A|v\rangle)^\dagger = \langle v|A^\dagger$.
   (c) Give an expression for $(|v\rangle\langle w|)^\dagger$ that does not use the $\dagger$ operation.
   (d) Prove that
   $$\left(\sum_{j=1}^{n} c_j A_j\right)^\dagger = \sum_{j=1}^{n} c_j^* A_j^\dagger.$$
   You can use the fact that $(A+B)^\dagger = A^\dagger + B^\dagger$.
   (e) The outer-bracket representation of A in basis $|\phi_1\rangle, \ldots, |\phi_N\rangle$ is
   $$\sum_{j=1}^{N}\sum_{k=1}^{N} \langle\phi_j|A|\phi_k\rangle|\phi_j\rangle\langle\phi_k|.$$
   Give an expression for $A^\dagger$ in outer-bracket notation using the same basis. Your expression should not use the $\dagger$ operation.
   (f) Let $|v_1\rangle, \ldots, |v_N\rangle$ set of eigenvectors for A with eigenvalues $\lambda_1, \ldots, \lambda_N$. In class we saw that the uter-bracket representation of A in basis $|v_1\rangle, \ldots, |v_N\rangle$ is
   $$\sum_{j=1}^{N} \lambda_j|\phi_j\rangle\langle\phi_j|.$$
   Give an expression for $A^\dagger$ in outer-bracket notation using the λ's and the $|v\rangle$'s. Your expression should not use the $\dagger$ operation.

For graduate students: skip problems 3-6, and do the following problem:

9. Prove that the eigenvalues of a unitary operator can be written in the form $e^{i\theta}$ for some real θ.

10. Prove that two eigenvectors of a Hermitian matrix with different eigenvalues are orthogonal. A Hermetian matrix has real eigenvalues. You can also used the fact that if A is Hermetian, then $A^\dagger = A$.

11. The Hadamard operator on one qubit may be written as

$$H = \frac{1}{\sqrt{2}} \left[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1| \right].$$

Give a closed form expression for $H^{\otimes n}$ using outer-bracket notation in the standard basis. (Hint: You will need to use the *dot product* of two strings. If x and y are two n-bit strings, then $x \cdot y = \sum_{j=1}^{n} x_j \cdot y_j$, where x_j is the j^{th} bit of x and y_j is the j^{th} bit of y.)