CS 190/264: Quantum Computation *Instructor: Sandy Irani*

Homework 1

Due: October 10, 2018

Note: graduate students are not required to do problems 1, 2, and 4, but are required to probelm 9. Undergraduates are required to do problems 1-8.

1. Let α and β be two complex numbers.
   (a) Prove that $(\alpha\beta)^* = \alpha^*\beta^*$.
   (b) Prove that $(\alpha + \beta)^* = \alpha^* + \beta^*$.

2. Let $\alpha = 3e^{i\pi 2/3}$ and $\beta = 2e^{-i\pi/4}$.
   (a) What is $|\alpha|$?
   (b) What is α^*?
   (c) Express α in standard $(a + bi)$ form.
   (d) What is $\alpha \cdot \beta$?
   (e) What is $\alpha^* \cdot \beta$?

3. Define

$$|\phi\rangle = \frac{\sqrt{3}}{4}|000\rangle + \frac{1}{4}|001\rangle + \frac{1}{2\sqrt{2}}|010\rangle + \frac{1}{2\sqrt{2}}|011\rangle + \frac{1}{2\sqrt{2}}|100\rangle + \frac{1}{2\sqrt{2}}|101\rangle + \frac{1}{2\sqrt{2}}|110\rangle + \frac{1}{2\sqrt{2}}|111\rangle.$$

   (a) Suppose that all three qubits are measured. What is the probability that the outcome of the measurement is 001?
   (b) If the outcome of the measurement is 001, then what is the state after measurement?
   (c) Consider the state $|\phi\rangle$ again before measurement. Suppose now that only the last bit is measured. What's the probability that the outcome of the measurement is 1?
   (d) If the last bit is measured and the outcome is 1, then what is the state after measurement?

4. Consider the state of a single qubit

$$|\phi\rangle = \frac{e^{i\pi/6}}{\sqrt{3}}|0\rangle + \frac{\sqrt{2}e^{-i\pi/4}}{\sqrt{3}}|1\rangle.$$

   If the qubit is measured, what's the probability that the outcome is 0?

5. Define the states $|\phi\rangle$ and $|\psi\rangle$ as:

$$|\phi\rangle = \begin{pmatrix} 1/\sqrt{3} \\ (1+i)/\sqrt{6} \\ 0 \\ i/\sqrt{3} \end{pmatrix} \quad \text{and} \quad |\psi\rangle = \begin{pmatrix} i/2 \\ (1-i)/4 \\ (1+i)/4 \\ 1/2 \end{pmatrix}.$$

   (a) Express $\langle\psi|$ as a row or column vector.

   (b) Express $\frac{\sqrt{3}}{2}|\psi\rangle + \frac{1}{4}|\phi\rangle$ as a row or column vector.

   (c) Calculate $\langle\phi|\psi\rangle$.

   (d) What is the L_2 norm of $|\psi\rangle$?

6. Suppose that $|\phi\rangle \in \mathbb{C}^N$. Prove that $\langle\phi|\phi\rangle$ is always a real non-negative number.

7. Consider the following four states of a 2-qubit system:

   - $|\phi_0\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$
   - $|\phi_1\rangle = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$
   - $|\phi_2\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$
   - $|\phi_3\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$

   (a) Do the states $|\phi_0\rangle$, $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ form an orthonormal basis of the Hilbert space spanned by the 2-qubit system?

   (b) Express $|01\rangle$ as a linear combination of the $|\phi_j\rangle$'s.

   (c) Express the following linear operator in matrix form: $|\phi_2\rangle\langle\phi_0|$,

   (d) Without creating the matrix, determine how the operator $|\phi_3\rangle\langle\phi_2|$ acts on a state

$$|\phi\rangle = \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}.$$

   That is, express the result of the operator $|\phi_3\rangle\langle\phi_2|$ acting on $|\phi\rangle$ as a function of the α's.

8. Recall the definitions for $|+\rangle$ and $|-\rangle$ which form an orhonormal basis of $\mathbb{C}^2$:

$$|+\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \text{and} \quad |-\rangle = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}.$$

   Express the following linear operator using outer-bracket notation and the $|+\rangle, |-\rangle$ basis:

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

For graduate students: skip problems 1, 2, and 4, and do the following problem:

9. Let A be a linear operator acting on $\mathbb{C}^N$. $|v_1\rangle, \ldots, |v_N\rangle$ and $|w_1\rangle, \ldots, |w_N\rangle$ are two different ortho-normal bases of $\mathbb{C}^N$. We will define two $N \times N$ matrics A' and A''. The elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''. In other words, describe a way to transform matrix A' into the matrix A''.