VOTE DELEGATION IN A VOTING WITH WEIGHTED MAJORITY RULE

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Abstract

We examine vote delegation in a dichotomous voting profile where a correct alternative exists, each voter supports each alternative independently with a fixed probability, and delegators do not know the preferences of representatives. We are interested in investigating whether free delegation favors the correct alternative, who has a higher chance of winning ex-ante. We study this question in a weighted voting system with voters and delegators having arbitrary weight distributions. By defining a dictator as a voter whose weight is greater than the sum of other voters' weights, we prove that free delegation favors the correct alternative if a dictator exists among the voters. Moreover, we prove that the only weight distribution of voters that minimizes the likelihood of the correct collective decision is the case where a dictator exists in the system. Furthermore, we provide bounds for the change in the probability of correct collective decision when there is only one delegator.

1 Introduction

This thesis is on extensions of Liquid democracy, which is a proxy voting method where proxies are delegable. This decision-making protocol lies between representative democracy and direct democracy and could be a remedy to the problem of preference bias in representative democracies and the problem of information aggregation in direct democracies. Throughout this thesis, we assume that there is voting between two alternatives, where a correct alternative exists, and all voters receive the correct signal with a fixed probability higher than 0.5. A recent study by Gersbach et al. in [1] has revealed that regardless of the number of voters and delegators, free delegation favors the wrong alternative under the following assumptions:

1. Each voter casts one vote; thereby, all voters have the same weight.
2. All delegators have the same weight.
3. Delegators do not know the representatives' preferences and delegate randomly.

This thesis extends the previous research by investigating whether the same result holds if the first and the second assumptions are violated. In section 4, we try to answer the following research questions:

1. *In a weighted voting system, can we claim that in the general case where voters and delegators have arbitrary weight distributions, delegation favors the wrong alternative?*
2. *If the answer to the previous question is no, can we propose some metrics for evaluating weights to assess whether delegation works in favor of the correct alternative?*

These questions often arise in the context of blockchains and proof-of-stake protocols, where individuals can delegate their stake to other individuals to participate in the rewards from validating transactions without running the validation software themselves.

In these systems, each person's weight is determined based on their stakes, and thus both voters and delegators can have arbitrary weights. The answer to the questions above is vital for the blockchain community[1]. Suppose delegation in some cases favors the wrong alternative, then the chance of having a malicious attack by them increases if delegation is allowed. Therefore the blockchain community may become more conservative towards delegation in those situations.

In the second half of this thesis, we use a different approach by removing the random delegation assumption. We allow the blockchain protocol to decide whom to allocate the delegated votes to maximize the probability of the correct collective decision. Solving this optimization problem is hard because all the subsets should be checked. If there are less than five voters in the system, the probability of the correct collective decision is maximized when the delegated vote is allocated to the weakest voter. However, if there are more voters, even in the case of five voters, allocation to the weakest voter is not necessarily the optimal solution anymore. In fact, the decision problem of whom to allocate the delegated votes to maximize the probability of the correct collective decision may not even be NP-complete because there is no obvious short polynomial certificate for it. That is why in section 5, we study an easier problem: the maximum change in the probability of the correct collective decision through delegation of one vote. In this part, we provide some information about the highest increase in the probability of the correct collective decision if ideally we knew whom to allocate the delegated vote to maximize the probability of the correct collective decision.

In section 2 we review related literature. In section 3, we describe our model assumptions. In section 4.1, we provide a counterexample to show that the answer to the first question discussed above is *No*. The importance of the second question comes from the fact that it is computationally expensive to find the probability of the correct collective

[1] See [2], and [3] which are the first two papers that offered delegation protocol for the proof-of-stake blockchains.

decision for an arbitrary distribution of voters' weights. Thereby, calculating and comparing the probabilities of the correct collective decision before and after the delegation is expensive. If we find sufficient conditions on weight distributions that help us identify which group benefits from the delegation, we can sometimes circumvent solving a computationally expensive problem. In section 4.2, we propose one of these sufficient conditions, and in section 4.3, we propose a more general condition. Section 4.4 contains some simulations that support our result. In section 5, we provide bounds for the change in the probability of correct collective decision when there is only one delegator. Finally, in section 6 we conclude by summarizing the main findings of this thesis and bringing up our conjectures that need further investigations in future research.

2 Related Literature

In this thesis, we use a similar setting as used in [4]; namely, we assume that a correct alternative exists and all voters aim to find him/her and vote for him/her based on the random signal they receive. The famous Condorcet's Jury theorem in [4] states that the probability of the correct collective decision goes to one if the probability of receiving the correct signal is higher than 0.5, the number of voters converges to infinity, and the decision rule is simple majority voting. In this thesis, we study a weighted voting setting, and instead of focusing on the absolute probability number, we focus on the relative probability values before and after the delegation. In other words, this thesis studies the change in the probability of the correct collective decision through delegation.

Gersbach et al. in [1] have proven that delegation decreases the probability of the correct collective decision if everyone has the same voting weight. In other words, the delegation turns a perfectly balanced weight distribution of voters into an unbalanced one, which is not in favor of the correct alternative. We have proven that delegation increases the probability of the correct collective decision in a perfectly unbalanced setting where a dictator determines the final outcome. In other words, the delegation most probably turns a perfectly unbalanced weight distribution into a more balanced one, which favors the correct alternative. The results we have obtained support the main spirit in [1], that is, moving towards a more balanced system favors the correct alternative. Therefore, whenever a delegation turns the voters' weight distribution into a more balanced format, it increases the chance of the correct collective decision, and it has the opposite effect when it makes the system more unbalanced.

This thesis and [5] both study a weighted voting system with two alternatives where one alternative is the correct one. The results of both studies are similar in spirit. In [5], the authors provide the weight distribution of the voters that maximizes the probability of the correct collective decision, while this thesis provides the weight distribution of the voters that minimizes the probability of the correct collective decision. In the setting where all voters receive the correct signal with the same probability, authors of [5] have shown that equal weights maximizes the probability of the correct collective decision. This result is a complement to one of our results which shows dictatorship minimizes the probability of the correct collective decision.

In [6], authors have shown that local delegation to more competent voters will not necessarily improve the likelihood of the correct collective decision. Our setting differs from [6] in the sense that everyone is equally competent, and the delegation is not based on competency; rather, it is random. Despite these differences, our work and [6] both show that vote delegation does not necessarily favor the correct candidate. In fact, initial weight distribution and the network structure(structure of the random delegation ex-post) matter in determining the effect of delegation.

A couple of recent papers studied vote delegation and liquid democracy from a game-theoretic perspective. Authors of [7] have shown that delegation games with deterministic type profiles or effortless voting always have a Nash Equilibrium. Authors of [8] have studied a system where iterative delegation is allowed and have shown that it is NP-Complete to decide whether a Nash equilibrium exists. However, we use a vote delegation framework where everyone has the same competencies, iterative delegation is not allowed, delegators abstain in traditional voting, and delegators do not know the voters' preferences and delegate their votes randomly.

In [9], the author developed a probabilistic approach to the social choice problem in 1973. We use a similar framework by assigning each voter a fixed probability of receiving the correct signal. Similar to [9], we assume that the actual choice is made by a random device using social probabilities derived from voter probability of receiving the correct signal.

In [10], authors found a strong correlation between voter equity and collective irrationality for the decision rules that never produce a social tie. In this thesis, we found a strong correlation between voter equity and the probability of the correct collective decision. In [10], variance is used as a measure of voter equity. In this thesis, we use variance as well as the highest Shapley-Shubik power index and the Gini coefficient as a measure of voter equity. In 4.4 we show that all three measures of voter equity have a strong correlation with the probability of the correct collective decision.

In [11], authors analyzed a setting where a certain number of misbehaving voters exist. Gersbach et al. in [11] studied the relationship between the change in the probability of the correct collective decision through delegation and the number of misbehaving voters. However, misbehaving voters do not exist in our setting since every voter that receives the correct signal votes for the correct alternative. Hence, this thesis studies the relationship between the change in the probability of the correct collective decision through delegation and the initial weight distribution of voters and delegators.

3 Model

We use the same framework as the one proposed in [1]. A group of individuals needs to decide between two alternatives, A and B. We assume one correct alternative exists, namely A, in our setting. All voters aim to find the correct alternative between the two. All individuals receive the signal of the correct alternative independently and with fixed probability p where $(0 < p < 1)$. We call all the voters who vote for A "A-voters" and those who vote for B "B-voters". In this thesis, we assume that $p > 0.5$, thereby, with high probability, the majority receives the correct signal and consequently votes in favor of the correct alternative. In other words, A-voters are the majority. Some individuals do not want to participate in the voting. In traditional voting, such people would abstain, but in liquid democracy, these people, who are called delegators, delegate their vote to let others decide on their behalf. Therefore, the number of voters in traditional voting and liquid democracy stays the same; only the weight of each voter might change. Vote delegation creates a weighted voting setting since one person's vote might be counted several times because that person is the representative of all the people who had delegated their votes to him/her. We assume even a more general case and allow the voters to have different weights even in traditional voting. Moreover, we define a dictator as a voter whose weight is greater than the sum of other voters' weights.

The voting rule is a simple majority voting, in which people who receive the correct signal vote for A, and others vote for B. Everyone's vote is counted according to their weight. Voters' weights are positive and integer numbers.[2] N represents a set that contains all voters' indexes. **w'(s)** is a function defined over subset space that receives a subset of indexes as an input and outputs the total weight of the subset members before delegation. **w(s)** is a function that receives a subset of indexes as an input and outputs the total weight of the subset members after delegation. The probability of the correct collective decision before delegation(traditional voting) is denoted as $P(p)$ and

[2] If the weights are rational but not integer, all weights can be multiplied to a fixed number to obtain integer weights. In other words, only the relative weight values are important.

can be expressed as the following:

$$P(p) = \sum_{R \subseteq N, w'(R) > w'(N)/2} p^{|R|}(1-p)^{n-|R|}.$$

The probability of correct collective decision after delegation(liquid democracy framework) is denoted as $P_{deleg}(p)$ and can be expressed as the following:

$$P_{deleg}(p) = \sum_{R \subseteq N, w(R) > w(N)/2} p^{|R|}(1-p)^{n-|R|}.$$

4 Many Delegators

In this section, we study the effect of delegation on the probability of the correct collective decision in a setting where there are many delegators. First, we show that in contrast to the case studied in [1], there are cases where the probability of the correct collective decision increases after the delegation. Then we introduce a special case in which delegation increases the probability of the correct collective decision. Then, we generalize and prove that delegation always improves the chance of correct collective decision in the case of dictatorship.

4.1 Example: weight distribution matters

Consider having three voters with weights $(1, 1, 5)$ and only one delegator with weight 4. The probability that A wins in the conventional case is the following:

$$P(p) = p^3 + p(1-p)^2 + 2p^2(1-p) = p.$$

After delegation, the probability of correct collective decision can be described as the following:

$$P_{deleg}(p) = \begin{cases} p^3 + p(1-p)^2 + 2p^2(1-p) & \text{with probability } \dfrac{1}{3} \text{ new weights are } (1,1,9), \\ 3p^2(1-p) + p^3 & \text{with probability } \dfrac{2}{3} \text{ new weights are } (1,5,5). \end{cases} \quad (1)$$

In the first case, the probability of the correct collective decision is similar to the conventional case. However, in the second case, when $p > 0.5$, $P_{deleg} > P(p)$. The same result is obtained if the delegator's weight is 3 or 5. This example shows that delegation may sometimes favor the correct alternative by increasing the winning probability. Therefore, it is impossible to make a uniform claim about which candidate benefits from the delegation, and we need to study this question for different weight distributions separately.

4.2 Special dictatorship

This part examines a special setting which is called **special dictatorship**.

Definition 1. *special dictatorship is a case where the following conditions hold:*

1. *All voters except for one have the same weight.*
2. *One dictator exists among the voters.*
3. *One delegator with an arbitrary weight exists in the system.*

Weights can be normalized so that n voters each have weight 1, and the dictator has a weight M where $M > n$[3]. Every voter votes for A with some probability p, $(0.5 < p < 1)$. Also, there is only one delegator with weight M' where M' is an arbitrary integer number. We prove that, in this case, delegation favors the correct alternative.

Theorem 1. *In a special dictatorship,* $P_{deleg}(p) \geq P(p)$.

[3]Here we assume there are $n+1$ voters.

Proof.

$$P(p) = p\left[\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\right] = p.$$

$$P_{deleg}(p) = \begin{cases} p & \text{with probability } \dfrac{1}{n+1}, \\ P'(p) & \text{with probability } \dfrac{n}{n+1}. \end{cases} \tag{2}$$

In the first case, with probability of $\frac{1}{n+1}$, the delegator delegates to the dictator and makes him/her even more powerful. Therefore, the probability of correct collective decision is equal to the conventional case.

In the second case, the delegator delegates to one of the normal voters and changes his/her weight to $M' + 1$. For different values of M', $P'(p)$ has different expressions. Based on the value of M', one of the following cases may happen.

- $M' + n < M$ In this case there is still a dictator after the delegation. Hence, the probability of correct collective decision is equal to the conventional case.

- $M' + n = M$ In this case, the probability of correct collective decision is denoted by $P_1(p)$ and is obtained by the following equation(ties are broken arbitrarily).

$$P_1(p) = \frac{1}{2}p(1-p)^n + \frac{1}{2}p^n(1-p) + p\sum_{k=1}^{n}\binom{n}{k}p^k(1-p)^{n-k}.$$

$$P_1(p) - P(p) = 0.5(p^n(1-p) - p(1-p)^n).$$

  $P_1(p) - P(p)$ is always positive for $p > 0.5$, so $P_1(p) > P(p)$.

- $M' + n = M + j, j \geq 1$ **where** j **is odd:** In this case, the probability of correct collective decision is denoted by $P_2(p)$ and is obtained by the following equation.

$$P_2(p) = g_1(p) + g_2(p) + g_3(p).$$

$$\begin{cases} g_1(p) = p^2 \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}. \\ g_2(p) = p(1-p) \sum_{\lfloor \frac{j}{2} \rfloor +1}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}. \\ g_3(p) = p(1-p) \sum_{n-1-\lceil \frac{j}{2} \rceil}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}. \end{cases} \tag{3}$$

In $g_1(p)$ and $g_3(p)$, by change of variables we get the following equations:

$$g_1(p) = p^2 \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k} = \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k+1} (1-p)^{n-k}.$$

$$g_3(p) = \sum_{n-1-\lceil \frac{j}{2} \rceil}^{n-1} \binom{n-1}{k} p^{k+1} (1-p)^{n-k} = \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor +1} \binom{n-1}{k} p^{n-k} (1-p)^{k+1}.$$

So $P_2(p)$ can be rewritten as:

$$\begin{aligned} P_2(p) = & \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k+1} (1-p)^{n-k} + \sum_{k=0}^{\lfloor \frac{j}{2} \rfloor +1} \binom{n-1}{k} p^{n-k} (1-p)^{k+1} \\ & + \sum_{k=\lfloor \frac{j}{2} \rfloor +1}^{n-1} \binom{n-1}{k} p^{k+1} (1-p)^{n-k}. \end{aligned} \tag{4}$$

In the equation for $P(p)$, we separate cases where $k = 0$ and $k = n$, and for the rest we use the following formula: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. The new equation of $P(p)$ is the following:

$$P(p) = p(1-p)^n + \sum_{k=1}^{n-1} \binom{n-1}{k} p^{k+1} (1-p)^{n-k} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} p^{k+1} (1-p)^{n-k} + p^{n+1}.$$

Now, we can write:

$$\begin{aligned} P_2(p) - P(p) = & \, g(j,p) + p^n(1-p) + \binom{n-1}{\lfloor \frac{j}{2} \rfloor + 1} p^{n-1-\lfloor \frac{j}{2} \rfloor} (1-p)^{\lfloor \frac{j}{2} \rfloor} \\ & + p(1-p)[p^{n-1} - (1-p)^{n-1}]. \end{aligned} \tag{5}$$

$$g(j,p) = \sum_{k=1}^{\lfloor \frac{j}{2} \rfloor} \binom{n-1}{k} [p^{n-k}(1-p)^{k+1} - p^{k+1}(1-p)^{n-k}].$$

The last three phrases of the right hand side expression are always positive when $p > 0.5$. Now we prove that even if $n - 1 < j$, the first phrase is non-negative. In this case we can write the following:

$$
\begin{aligned}
g(j,p) = &\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1}{k} [p^{n-k}(1-p)^{k+1} - p^{k+1}(1-p)^{n-k}] \\
&+ \sum_{k=\lfloor\frac{n-1}{2}\rfloor+1}^{min(n-1,\lfloor\frac{j}{2}\rfloor)} \binom{n-1}{k} [p^{n-k}(1-p)^{k+1} - p^{k+1}(1-p)^{n-k}].
\end{aligned} \tag{6}
$$

After a change of parameter in the second sum of the right hand side phrase we get:

$$
\begin{aligned}
&\sum_{k=\lfloor\frac{n-1}{2}\rfloor+1}^{min(n-1,\lfloor\frac{j}{2}\rfloor)} \binom{n-1}{k} [p^{n-k}(1-p)^{k+1} - p^{k+1}(1-p)^{n-k}] \\
&= \sum_{k=max(0,n-1-\lfloor\frac{j}{2}\rfloor)}^{\lfloor\frac{n-1}{2}\rfloor+1} \binom{n-1}{k} [p^{k+1}(1-p)^{n-k} - p^{n-k}(1-p)^{k+1}].
\end{aligned} \tag{7}
$$

So $g(j,p)$ can be written as:

$$
\begin{aligned}
g(j,p) = &\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-1}{k} [p^{n-k}(1-p)^{k+1} - p^{k+1}(1-p)^{n-k}] \\
&+ \sum_{k=max(0,n-1-\lfloor\frac{j}{2}\rfloor)}^{\lfloor\frac{n-1}{2}\rfloor+1} \binom{n-1}{k} [p^{k+1}(1-p)^{n-k} - p^{n-k}(1-p)^{k+1}].
\end{aligned} \tag{8}
$$

In equation 8, some terms in the second sum cancel out with some terms in the first sum, and the remaining terms of the first sum are all positive for $p > 0.5$. for $k = \lfloor\frac{n-1}{2}\rfloor + 1$, it can be easily seen that the following expression is positive:

$$
\binom{n-1}{\lfloor\frac{n-1}{2}\rfloor + 1} [p^{\lfloor\frac{n-1}{2}\rfloor+2}(1-p)^{n-\lfloor\frac{n-1}{2}\rfloor-1} - p^{n-\lfloor\frac{n-1}{2}\rfloor-1}(1-p)^{\lfloor\frac{n-1}{2}\rfloor+2}].
$$

Finally, for $k = 0$ (only if $n - 1 < \lfloor\frac{j}{2}\rfloor$), we get $p(1-p)^n - p^n(1-p)$, where $-p^n(1-p)$ is cancelled out with the $p^n(1-p)$ term in $P_2(p) - P(p)$. Therefore for $p > 0.5$, $P_2(p) > P(p)$.

$\bullet M' + n = M + j, j \geq 1$ **where** j **is even:** In this case, the probability of correct collective decision is denoted by $P_3(p)$ and is obtained by the following equation.

$$P_3(p) = g_1(p) + g_2(p) + g_3(p) + g_4(p) + g_5(p).$$

$$\begin{cases} g_1(p) = p^2 \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}. \\ g_2(p) = p(1-p) \sum_{\lfloor \frac{j}{2} \rfloor + 1}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}. \\ g_3(p) = p(1-p) \sum_{n-1-\lceil \frac{j}{2} \rceil}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}. \\ g_4(p) = \frac{1}{2} p(1-p) \binom{n-1}{\frac{j}{2}} p^{\frac{j}{2}} (1-p)^{n-1-\frac{j}{2}}. \\ g_5(p) = \frac{1}{2} p(1-p) \binom{n-1}{n-1-\frac{j}{2}} p^{n-1-\frac{j}{2}} (1-p)^{\frac{j}{2}}. \end{cases} \tag{9}$$

It can be easily seen that $P_3(p) > P_2(p)$ because $P_3(p)$ has all the phrases of $P_2(p)$ plus two non-negative probabilities. Since we proved that $P_2(p) > P(p)$, it is straightforward that $P_3(p) > P(p)$.

□

4.3 General dictatorship

This part provides one sufficient condition on the voters' weight distribution that guarantees the benefit of delegation to the correct alternative. We prove that if a dictator exists among the voters, delegation increases the probability of correct collective decision. In other words, the probability of the correct collective decision is minimized in the case of a dictatorship(where a dictator exists). To study this case, consider having n voters with weights w_1 to w_n where for all i, j if $i < j$, $w_i \leq w_j$ and $w_n > \sum_{i=1}^{n-1} w_i$.

Theorem 2. *Dictatorship minimizes the probability of the correct collective decision.*

Proof. The theorem above is equivalent to the following: If there is a dictator among the voters, $P_{deleg}(p) \geq P(p)$. In other words, the delegation cannot worsen the situation for the correct alternative because having a dictator is the worst possible scenario. Since there is a dictator before delegation, the probability of the correct collective decision in the conventional case is $P(p) = p$. If there is a dictator even after the delegation, the probability of the correct collective decision stays the same as the conventional case. If there is no dictator after the delegation, the probability of the correct collective decision is different from the conventional case, and the new probability can be formulated as below:

$$P_{deleg}(p) = \sum_{s \subseteq N, w(s) > w(N)/2} p^{|s|}(1-p)^{n-|s|}.$$

We can separate the subsets containing n(the dictator in the conventional case), denoted by S, and the ones that do not contain n, denoted by R, then we can rewrite $P_{deleg}(p)$ in the following way:

$$P_{deleg}(p) = \sum_{s \subseteq N-1, w(s \cup \{n\}) > w(N)/2} p^{|s|+1}(1-p)^{n-|s|-1} + \sum_{R \subseteq N-1, w(R) > w(N)/2} p^{|R|}(1-p)^{n-|R|}.$$

$$P(p) = p = \sum_{s \subseteq N-1} p^{|s|+1}(1-p)^{n-|s|-1}.$$

$N-1$ represents a set that contains all voters' indexes except the nth person. In other words, $N-1 = N \setminus \{1\}$. All terms in the left summand of $P_{deleg}(p)$ cancel out with some terms in $P(p)$. So we can write:

$$P_{deleg}(p) = \sum_{R \subseteq N-1, w(R) > w(N)/2} p^{|R|}(1-p)^{n-|R|}.$$

$$P(p) = \sum_{s \subseteq N-1, w(s \cup \{n\}) < w(N)/2} p^{|s|+1}(1-p)^{n-|s|-1}.$$

For every subset R in $P_{deleg}(p)$ there exists a complementary subset, namely $\bar{R}$ in $P(p)$, where $|R|+|\bar{R}|= n$. Also, we can separate the cases where the size of the subsets is greater than, or less than $n/2$. Thereby, we can write the following formula:

$$P_{deleg}(p) - P(p) = \sum_{R \subseteq N-1, w(R) > w(N)/2, |R| < n/2} G(p) + \sum_{R \subseteq N-1, w(R) > w(N)/2, |R| > n/2} G(p),$$

where

$$G(p) = p^{|R|}(1-p)^{n-|R|} - p^{|\bar{R}|+1}(1-p)^{n-|\bar{R}|-1}.$$

It can be easily seen that for any subset R that satisfies the conditions $R \subseteq N - 1, w(R) > w(N)/2, |R| \leq n/2$, there is at least one matching subset R' where $R' \subseteq N - 1, w(R') > w(N)/2, |R'| > n/2$[4]. Thereby, the number of subsets R where $R \subseteq N - 1, w(R) > w(N)/2, |R| \leq n/2$ is less than the number of subsets R' where $R' \subseteq N - 1, w(R') > w(N)/2, |R'| > n/2$. Thereby, for any $j < n/2$, where j is the size of R, the following expression exists in $P_{deleg}(p) - P(p)$:

$$\sum_{R \subseteq N-1, w(R) > w(N)/2, |R| = j} G_1(j,p) + \sum_{R \subseteq N-1, w(R) > w(N)/2, |R| = n-j} G_2(j,p),$$

where

$$G_1(j,p) = p^j(1-p)^{n-j} - p^{n-j+1}(1-p)^{j-1},$$

$$G_2(j,p) = p^{n-j}(1-p)^j - p^{j+1}(1-p)^{n-j-1}.$$

For j from 1 to $(n-1)/2$, we assume k_j is the number of winning subsets of $N - 1$ with size j. Hence, we can write the formulas below:

$$\begin{aligned} P_{deleg}(p) - P(p) = & k_1(p^1(1-p)^{n-1} - p^n) + (k_2 + k_2')(p^{n-1}(1-p) - p^2(1-p)^{n-2}) + \\ & k_{n-1/2}(p^{n-1/2}(1-p)^{n+1/2} - p^{n+3/2}(1-p)^{n-3/2}) + \\ & k(p^{n+1/2}(1-p)^{n-1/2} - p^{n+1/2}(1-p)^{n-1/2}) + \\ & \sum_{j=2}^{\frac{n-1}{2}-1} k_j(p^j(1-p)^{n-j} - p^{n-j+1}(1-p)^{j-1}) + \\ & (k_{j+1} + k_{j+1}')(p^{n-j}(1-p)^j - p^{j+1}(1-p)^{n-j-1}). \end{aligned} \tag{10}$$

Hence we can write,

$$P_{deleg}(p) - P(p) = k_1(p^1(1-p)^{n-1} - p^n) + \sum_{j=2}^{n-1/2} k_j'(p^{n-j+1}(1-p)^{j-1} - p^j(1-p)^{n-j}). \tag{11}$$

[4]It is enough to copy all elements of R into R' and select the remaining $|R'| - |R|$ elements from the remaining elements randomly.

k_1 represents the number of subsets with size 1 that satisfy the following conditions: $\{R \subseteq N-1, w(R) > w(N)/2\}$. We have analyzed the case where a dictator exists in the new system, previously. Here, we are only focusing on the cases with no dictator; hence, $k_1 = 0$. We have discussed why the number of subsets R where $|R| > n/2$ is greater than the number of subsets R where $|R| < n/2$. Therefore all k'_js are positive. The multiplier for each k'_j is $G_2(j-1, p)$ which is always positive for $p > 0.5$ and $j < n/2$. Therefore $P_{deleg}(p) - P(p) > 0$. □

This result shows that any weight distribution of voters that does not contain a dictator is better for the correct alternative than having a dictator in the system. Thereby, regardless of the number of delegators and their weight distribution, delegation favors the correct alternative by breaking up the monopoly of the dictator. To put it differently, the only weight distribution of voters that minimizes the likelihood of the correct collective decision is the case where a dictator exists in the system.

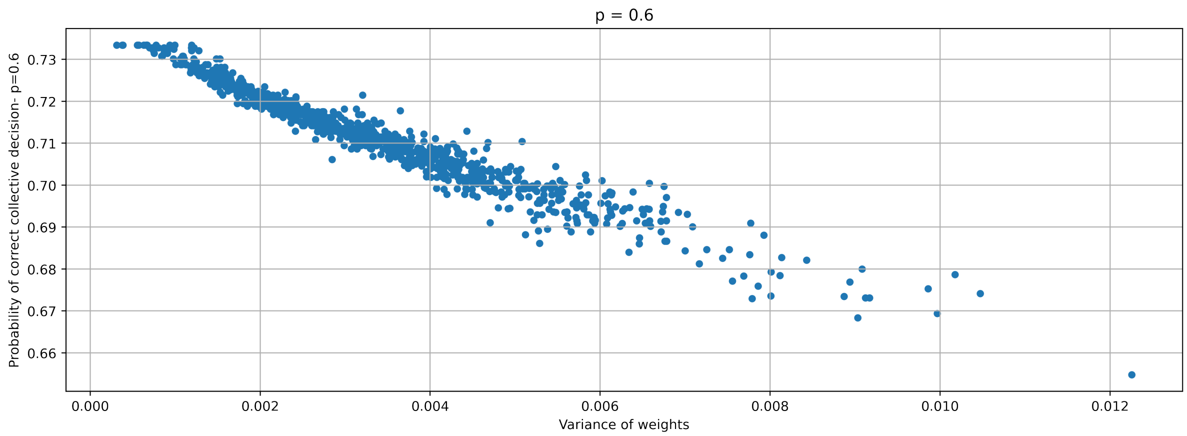
4.4 Simulation

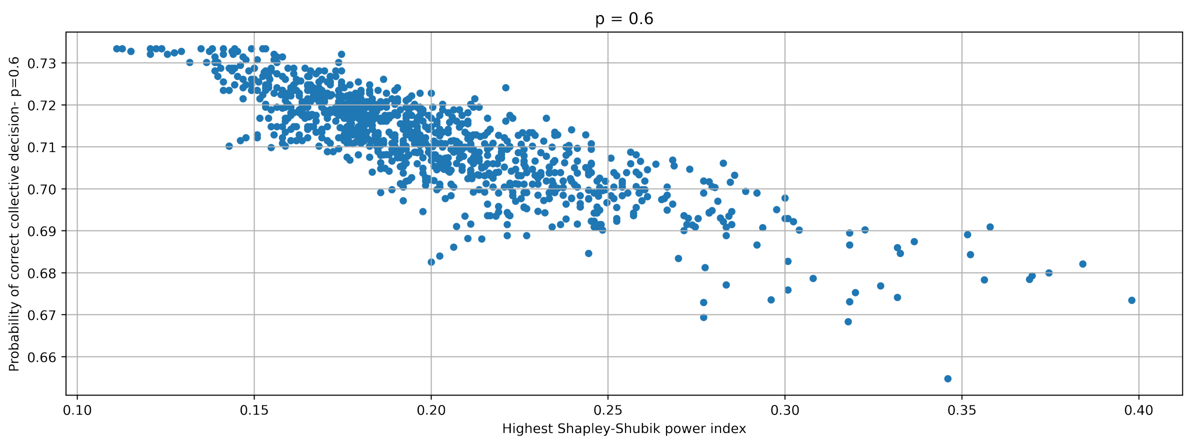
Gersbach et al. in [1] studied an extremely balanced weight distribution, and this thesis studied an extremely unbalanced distribution so far. However, it is interesting to know whether we can generalize the results we obtained and claim that delegation improves the chance of the correct collective decision if it pushes the system towards a more balanced weight distribution. To examine this question, it is crucial to elaborate on what we mean by saying a system is more balanced than the other and how to measure balance.

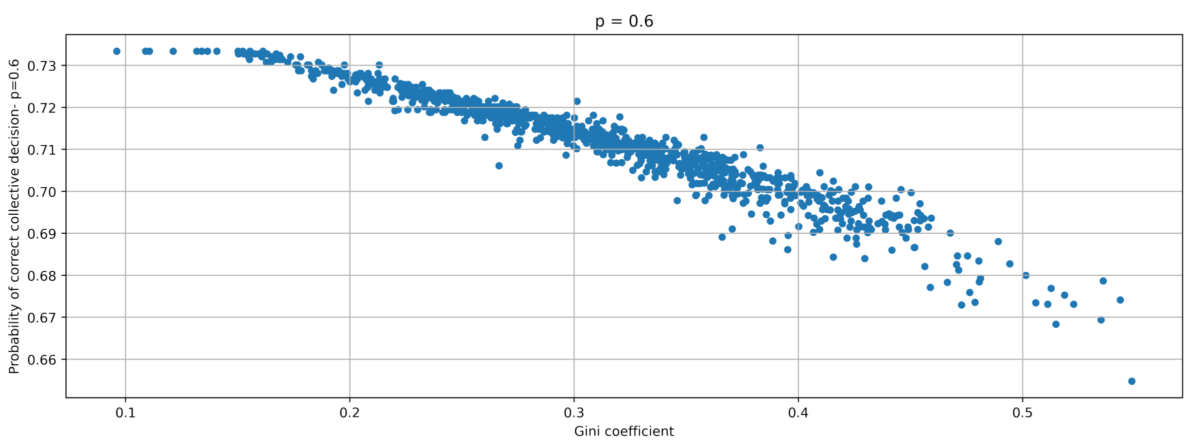
Through the following simulations, we evaluated how the probability of the correct collective decision changes with respect to variance, Gini coefficient, and the maximum of the Shapley-Shubik power indexes[12][5]. We simulated 10 voters for 1000 times. We

[5]We used the powerindex python package provided in https://github.com/maxlit/powerindex to compute the Shapley-Shubik indexes.

attributed normalized random values to their weights each time so that weights summed up to 1. Then, we calculated the probability of the correct collective decision, variance, the Gini coefficient, and the maximum of Shapley-Shubik power indexes in each case. In the plots below, the results are shown.







The following table contains the correlation between the variance, the highest Shapley-Shubik power index, and the Gini coefficient with the probability of the correct collective decision.

.	Variance	Highest Shapley-Shubik index	Gini
Probability	-0.972	-0.843	-0.966

As we expected, the variance, the highest Shapley-Shubik power index, and the Gini coefficient all strongly correlate with the probability of the correct collective decision. However, as shown in the plots above, the probability of the correct collective decision is not monotonically decreasing in any of the cases. Thereby, none of the proposed measures can be used as a reliable measure for finding the sign of the probability change through delegation. Hence, we encourage the reader to study other ways to measure the balance of weight distributions and find better criteria on weight distributions that help us identify which group benefits from delegation.

5 One Delegator and Probability Change

In this section, the question of our interest is whether we can bound the greatest change in the probability of the correct collective decision after one random delegation. Such change is denoted as Δ and is defined as below:

$$\Delta = \sup_{W,w',p} (P_{deleg}(p) - P(p)),$$

where w' is the weight of the delegator and W is the vector of voters' weights before delegation where for all i, j if $i < j$, $w_i \leq w_j$. In this section, we use the vector of weights and indices interchangeably.

Definition 2. *The delegator randomly selects one of the voters and delegates his/her vote to him/her. This selected voter is called the* ***game-changer****.*

Definition 3. *If the following conditions hold, we call the case* ***one-dictator-to-two-dictators****:*

1. *There is a dictator in the initial system.*

2. *The delegator delegates his/her vote to one of the weak voters and makes the new weight of game-changer $w_n - \epsilon$, where ϵ is a very small number.*

3. *After the delegation, if the two powerful voters are on two different sides, the candidate with the higher number of proponents wins(majority voting).*

In the case of one-dictator-to-two-dictators, a dictator exists pre-delegation, and through the delegation process, his monopoly breaks. In the post-delegation system, there is no dictator anymore; instead, there are two voters with very high and similar voting power. Whenever these two powerful voters are on different sides, the power is balanced in a way that the candidate with more proponents(the majority) wins. Hence, if a candidate with a minority of supporters wants to win, it is not enough to have only one of the two powerful voters on his/her side; in fact, both powerful voters should be in the minority subset to form a winning subset.

Definition 4. *Changing-Minority-Winning-Subsets(k) or CMWS(k) is the set of subsets with size k that were winning pre-delegation and are losing post-delegation where* $k \leq \lceil \frac{n-2}{2} \rceil$.

Definition 5. *Changing-Minority-Winning-Subsets-Dictator-Included(k) or CMWSDI(k) is defined as the set of subsets with size k that were winning pre-delegation and are losing post-delegation, when a dictator exists, where* $k \leq \lceil \frac{n-2}{2} \rceil$.

From the definition above, it can be easily seen that all the minority subsets containing n and not the game-changer are in CMWSDI(k).

Definition 6. *Possible(k) is defined as the set of all the subsets of size k that do not contain the dictator or the game-changer, and can potentially be in the CMWS(k).*

For example, when $n = 9, k = 2$, $(7, 8)$ is in Possible(2), but $(6, 7)$ is not in Possible(2) because elements of 8 and 9 which are in the complementary subset dominate $(6, 7)$. In fact, for any n and $k = 2$, the only member of Possible(2) is $(n-2, n-1)$.

Definition 7. *A matching table is a table that for any $k \leq \lceil \frac{n-2}{2} \rceil$, and for any subset in Possible(k), provides a unique matching subset from CMWSDI(k).*

Theorem 3. *If a matching table exists for a given number of voters, maximum change in the probability of the correct collective decision(the tightest upper bound) through the delegation process is achieved in the case of one-dictator-to-two-dictators.*

Proof. To compare the probability of the correct collective decision across different cases, we need to compare the number of Changing-Minority-Winning-Subsets(CMWS) for different subset sizes. From the definition of CMWS(k), it can be easily seen that the "game-changer" should not belong to any of the subsets in CMWS(k). If there is a dictator in the pre-delegation phase, $CMWS(k) = CMWSDI(k)$ since all the minority winning subsets before delegation should contain the dictator. If there is another case where $|CMWS(k)|$ is greater than the case of one-dictator-to-two-dictators, CMWS(k) should contain at least one of the subsets in Possible(k). We show that the existence of any of such subsets in CMWS(k) means that at least one of the subsets in CMWSDI(k) cannot exist in CMWS(k) anymore. For example when $n = 9$, and $k = 2$, adding $(7, 8)$ to the CMWS(2), means that the following subsets in CMWSDI(k) cannot exist in CMWS(k) anymore: $\{(2, 9), (3, 9), (4, 9), (5, 9), (6, 9)\}$ for the following reason: If $(7, 8)$ is a pre-delegation winner, its complement is a pre-delegation loser. Therefore any subset of its complement is also a pre-delegation loser and cannot be in CMWS(k). If for any subset $s \in Possible(k)$, we find a unique subset $s' \in CMWSDI(k)$, where adding s to $CMWS(k)$ forces removing s' from CMWS(k), we conclude that $|CMWS(k)|$ for all $k \leq \lceil \frac{n-2}{2} \rceil$ is greatest when having one-dictator-to-two-dictators. Therefore, the existence of a matching table for a given n guarantees that the highest change in the probability of the correct alternative decision is achieved when having one-dictator-to-two-dictators.

□

Theorem 4. *In case of one-dictator-to-two-dictators, the change in the probability of the correct collective decision is upper-bounded by* 0.25*.*

Proof. Let us divide the minority winning subsets of the pre-delegation phase into two categories:

1. The subsets that contain both the dictator and the game-changer.

2. The subsets that contain the dictator but do not contain the game-changer.

After the delegation, the subsets of the first category are still the winning subsets, as the delegation has enforced their power. On the other hand, the subsets of the second category are the losing subsets after the delegation. This happens because these subsets are the minorities having only one of the powerful voters on their side. The greatest change in the probability of the correct collective decision is associated with the case where all of the subsets of the second category turn into losing subsets after delegation. Therefore, to compute the probability change, we need to compute the number of subsets in the second category, which equals $|CMWSDI(k)|$ for a given subset size k. Therefore, we can denote the probability change in the case of one-dictator-to-two-dictators as $\Delta P_{1:2}(p)$ and calculate it as the following:

$$\begin{aligned}\Delta P_{1:2}(p) &= \sum_{i=1}^{\frac{n-1}{2}} \binom{n-2}{i-1} [p^{n-i}(1-p)^{i} - p^{i}(1-p)^{n-i}] \\ &= p(1-p) \sum_{i=0}^{\frac{n-3}{2}} \binom{n-2}{i} [p^{n-2-i}(1-p)^{i} - p^{i}(1-p)^{n-2-i}] \\ &= p(1-p)(2P_A(n-2)-1),\end{aligned} \tag{12}$$

where $P_A(n)$ is the probability of the correct collective decision in a majority voting among n voters with equal weights. Since $P_A(n-2) \leq 1$, the following holds:

$$\begin{aligned}\Delta P_{1:2}(p) &= p(1-p)(2P_A(n-2)-1) \\ &\leq p(1-p) \\ &\leq 0.25.\end{aligned} \tag{13}$$

□

Corollary 1. *If a matching table exists for a given number of voters,* $\Delta \leq 0.25$.

Proof. Theorem 3 implies that the maximum change in the probability of the correct collective decision is upper-bounded by $\Delta P_{1:2}(p)$. Hence we can write:

$$P_{deleg}(p) - P(p) \leq \Delta P_{1:2}(p)$$

In theorem 4 we showed that

$$\Delta P_{1:2}(p) \leq 0.25.$$

Therefore we have the following:

$$P_{deleg}(p) - P(p) \leq 0.25$$

which directly implies:

$$\sup_{W,w',p} \left(P_{deleg}(p) - P(p)\right) = \Delta \leq 0.25.$$

□

5.1 Existence of a matching table

5.1.1 When $n \leq 9$, a matching table exists.

1. $n = 9$

   $Possible(1) = \{\emptyset\}$.

   $Possible(2) = \{(7,8)\}$.

   $Possible(3) = \{(6,7,8), (5,7,8), (4,7,8), (3,7,8), (2,7,8), (5,6,8), (5,6,7)\}$.

   $Possible(4) = \{(5,6,7,8), (4,6,7,8), (3,6,7,8), (2,6,7,8), (4,5,7,8), (3,5,7,8), (2,5,7,8), (3,4,7,8), (2,4,7,8), (2,3,7,8), (4,5,6,8), (3,5,6,8), (2,5,6,8), (3,4,6,8), (3,4,5,8), (4,5,6,7), (3,5,6,7), (2,5,6,7), (3,4,6,7), (3,4,5,7), (3,4,5,6)\}$.

In the following table, we show that for any of the subsets s above, there is a unique subset s' with size k that contains the dictator but does not contain the game-changer, where adding s to $CMWS(k)$ removes s' from CMWS(k). In other words, a matching table exists. In the following figure, the right table is the continuation of the left table.

s	s'
(7,8)	(9,2)
(6,7,8)	(9,4,5)
(5,7,8)	(9,2,6)
(4,7,8)	(9,3,6)
(3,7,8)	(9,4,6)
(2,7,8)	(9,5,6)
(5,6,8)	(9,4,7)
(5,6,7)	(9,4,8)
(5,6,7,8)	(9,2,3,4)
(4,6,7,8)	(9,2,3,5)
(3,6,7,8)	(9,2,4,5)
(2,6,7,8)	(9,3,4,5)
(4,5,7,8)	(9,2,3,6)
(3,5,7,8)	(9,2,4,6)
(2,5,7,8)	(9,3,4,6)

s	s'
(3,4,7,8)	(9,2,5,6)
(2,4,7,8)	(9,3,5,6)
(2,3,7,8)	(9,4,5,6)
(4,5,6,8)	(9,2,3,7)
(3,5,6,8)	(9,2,4,7)
(2,5,6,8)	(9,3,4,7)
(3,4,6,8)	(9,2,5,7)
(3,4,5,8)	(9,2,6,7)
(4,5,6,7)	(9,2,3,8)
(3,5,6,7)	(9,2,4,8)
(2,5,6,7)	(9,3,4,8)
(3,4,6,7)	(9,2,5,8)
(3,4,5,7)	(9,2,6,8)
(3,4,5,6)	(9,2,7,8)

2. $n = 8$

   $Possible(1) = \{\emptyset\}$.

   $Possible(2) = \{(6, 7)\}$.

   $Possible(3) = \{(5, 6, 7), (4, 6, 7), (3, 6, 7), (2, 6, 7), (4, 5, 7), (4, 5, 6)\}$.

   The following matching table exists:

s	s'
(6,7)	(8,2)
(5,6,7)	(8,3,4)
(4,6,7)	(8,2,3)
(3,6,7)	(8,2,5)
(2,6,7)	(8,4,5)
(4,5,7)	(8,3,6)
(4,5,6)	(8,3,7)

3. $n = 7$

   $Possible(1) = \{\emptyset\}$.

   $Possible(2) = \{(5,6)\}$.

   $Possible(3) = \{(4,5,6),(3,5,6),(2,5,6),(3,4,6),(3,4,5)\}$.

   The following matching table exists:

s	s'
(5,6)	(7,2)
(4,5,6)	(7,2,3)
(3,5,6)	(7,2,4)
(2,5,6)	(7,3,4)
(3,4,6)	(7,2,5)
(3,4,5)	(7,2,6)

4. $n = 6$

   $Possible(1) = \{\emptyset\}$.

   $Possible(2) = \{(4,5)\}$.

   The following matching table exists:

s	s'
(4,5)	(6,2)

5.1.2 Discussion of a general formula

If we can find a matching table for all values of n, maximum change in the probability of the correct collective decision can always be upper-bounded by 0.25. Proving the existence of a matching table requires a formula for finding a unique match from subsets of CMWSDI(k) for any of the subsets in Possible(k), for any $k \leq \lceil \frac{n-2}{2} \rceil$. Such a formula needs to guarantee that any subset in CMWSDI(k) is matched to at most one subset in Possible(k) and all subsets in Possible(k) are matched to one subset in CMWSDI(k), for any $k \leq \lceil \frac{n-2}{2} \rceil$. We provide this formula for special cases where $k = 2$ and $k = \lceil \frac{n-2}{2} \rceil$, and encourage the reader to think of formulas when $2 < k < \lceil \frac{n-2}{2} \rceil$.

1. When $k = 2$, since there is only one member in Possible(2), any subset in CMWSDI(2) that is a subset of the complement of subset in Possible(2) can be selected as its match. For example, $(n, n-3)$ can be matched to $(n-1, n-2)$.

2. When n is odd and $k = \lceil \frac{n-2}{2} \rceil$, a unique match for any subset in Possible($\lceil \frac{n-2}{2} \rceil$) is found by removing the game-changer from its complementary subset.

5.2 Lower-bounding Δ

Theorem 5. *For any number of voters, the case of one-dictator-to-two-dictators can be constructed.*

Proof. Consider that there are n voters, with the following initial weight distribution:

$$w_i = \begin{cases} 1 & 1 \leq i < n, \\ n^2 & i = n. \end{cases} \tag{14}$$

There is one delegator with weight $n^2 - \epsilon - 1$ where $\epsilon \leq 0.01$. If the delegator delegates his/her vote to one of the weak voters, the new weights become as follows:

$$w'_i = \begin{cases} 1 & 1 \le i < n-1, \\ n^2 - \epsilon & i = n-1, \\ n^2 & i = n. \end{cases} \tag{15}$$

After the delegation, all the winning minority subsets that contain the dictator but do not contain the game-changer turn into losing subsets because the two powerful voters balance each other on two sides, and the number of other voters determines the winner. Hence, this is a one-dictator-to-two-dictator case that can be constructed for any given number of voters(n). □

In theorem 4, we have shown that $\Delta P_{1:2}(p) = p(1-p)(2P_A(n-2)-1)$. We also have that $\Delta P_{1:2}(p) \le \sup_{W,w',p}(P_{deleg}(p) - P(p)) = \Delta$. Therefore, directly deduct the following:

$$p(1-p)(2P_A(n-2)-1) \le \Delta.$$

If $n \to \infty$, by central limit theorem we have that $P_A(n-2) \to 1$. If we also assume that p slowly converges to 0.5 from above, the change in the probability of the correct collective decision can reach 0.25 from below by arbitrary precision. Hence,

$$0.25 \le \Delta.$$

6 Conclusion

In this thesis, we studied the change in the probability of the correct collective decision through vote delegation in a weighted voting setting. First, we showed that delegation might increase or decrease the probability of the correct collective decision depending on the initial weight distribution of voters. Then, we described a special case where delegation favors the correct alternative. Then we generalized and proved that the probability of the correct collective decision is minimized when there is a dictator in the initial system. Therefore, in the case of dictatorship, delegation can only improve the situation for the correct alternative by increasing his/her chance of winning. The obtained results support the results of [1] by showing that moving towards a more balanced weight distribution favors the correct alternative. This thesis studied a perfectly unbalanced system where a dictator exists, and [1] studied a perfectly balanced system where everyone has the same vote weight. For the extreme cases studied in this thesis and [1], it is easy to see that any form of the delegation would turn the weight distributions into a less extreme format; therefore, whether the system is getting more balanced/unbalanced is easily observed. However, in other initial weight distributions, it might not be as obvious whether the system has gotten more balanced throughout the delegation. Therefore, it is crucial to have a standard definition for balance. An important topic for future research is to find a proper measure for balance and to study whether, based on that definition, moving towards a more balanced weight distribution favors the correct alternative. Ideally, if we can find a measure for balance that can be computed fast[6] and can monotonically increase the probability of the correct collective decision, we would be able to quickly determine whether, based on the initial weight distribution of the voters and the delegation proposals, delegation favors the correct alternative. By having such valuable information, the system can automatically adjust itself so that it would allow for delegation only in the cases where the chance of the correct alternative winning is improved[7].

In this thesis, we also studied the maximum amount of probability change when there are few voters and one delegator. We showed that when having few voters, the highest probability change is upper bounded by 0.25 and is associated with the case of one-dictator-to-two-dictators. We conjecture that for any number of voters, 0.25 is an upper

[6] In polynomial time.

[7] Since here we assume that the probability of receiving the correct signal is higher than 0.5, the candidate with more supporters is the correct candidate with high probability. Therefore, the system can assume that the correct candidate is the one that has the majority of the votes before the delegation.

bound for the change in the probability of the correct collective decision. In other words, we believe that we can find a matching table for all values of n. This conjecture should be examined in future research.

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