1 Asymptotic Notation

Recall that we say that $f(n)$ is $\mathcal{O}(g(n))$ (read: f of n is big-oh of g of n) if there is a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that for all $n > n_0$, $f(n) \leq cg(n)$.

1. Use this definition to show that $3n^2 + 12n + 1$ is $\mathcal{O}(n^2)$.

2. Find the $\mathcal{O}$-notation for the worst-case running time of the following algorithm. You do not need to, nor should you, give the leading constant or the n_0 value. Does your answer depend on whether or not the vector is sorted?

```
int linearSearch(const std::vector<int> & numbers, int target)
{
    int i;
    int n = numbers.size();
    for(i=0; i < n; i++)
    {
        if( numbers[i] == target )
        {
            return i;
        }
    }
    throw ElementNotFoundException("Element not found by linear search.");
}
```

3. The `find-min` problem is to, given a vector of comparable elements, to find the smallest element in the vector. F. Lake claims to have found an algorithm to solve `find-min` in an arbitrary vector of size n in better than $\mathcal{O}(n)$ time. Do you believe the claim? Why or why not?

4. Suppose we have two algorithms to solve the same problem; that problem has, as input, an array A of size n. Would it be better to have an algorithm that takes $20n$ operations or one that takes n^2 operations? Why?

2 Counting

5. A company has three new employees. There are 12 empty offices. How many different ways are there to assign offices to these employees such that they do not share offices?

6. How many ways are there to arrange 10 people in a line?

3 Discrete Probability

This is for discrete events. If each event has a uniform probability, then the probability an event happens is the number of ways it can happen divided by the number of ways something happens. This is sometimes written as "# yes / # total"

For example, if I roll a fair six sided die, what is the probability I roll a 3? That can happen one way, out of the six things that could happen, so 1/6.

7. Suppose I roll two standard, six-sided fair dice. What is the probability that the sum of two dice rolls is 7?

Expected Value. One way to think of this is the weighted average of possible outcomes. Here is a somewhat simplified version that is fine for ICS 46. Suppose we are going to play the following betting game. You pay me some amount of money and then roll a fair, six-sided die. I then pay you a number of dollars equal to the number of pips[1] showing at the top after the roll. What should I charge you so that we both "expect" (I am using that word deliberately, and I am being hand-wavy about it deliberately) the same return for this?

To answer this, ignore the fact that I am charging you up front. How many dollars do you think you'll get from me? One-sixth of the time, you'll get \$1. One-sixth, you'll get \$2, and so on. So your total is

$$\frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \ldots + \frac{1}{6} \times 6 = \frac{21}{6} = 3.5$$

That's the "expected value" of the roll. If I charge you \$3.50 to play this game, we both "expect" to break even. Equivalently, if I were to open a gambling house and have this game available, and I charged \$3.50 per person to play, I would "expect" to break even long-term.

Students who find that narrative interesting might enjoy reading more about the history of probability. There is a considerable amount of early discoveries that were financed by wealthy gamblers looking to have mathematicians provide them an edge in their hobby. Perhaps not coincidentally, a lot of gambling operations still do so.

This brings us to the **Linearity of Expectations**: $E(X_1 + X_2 + ... + X_n) = E(X_1) + E(X_2) + ... + E(X_n)$. Also, $E(aX + b) = aE(X) + b$.

Suppose we were going to play the same game as earlier, except you are going to roll *three* dice, and I am going to pay you the sum of the numbers shown. We could determine that there are 216 outcomes, of which there is one way you roll a total of three, three ways to roll a total of four, and so on...

$$\frac{1}{216} \times 1 + \frac{3}{216} \times 2 + \ldots$$

[1]That's the word for the dots on the dice.

In principle, you could write out that summation and calculate it (or better yet, write a short computer program to write it out and calculate it). But intuitively, you might suspect that this is no different than rolling three dice, one after the other, and collecting the money for each. Linearity of expectation is saying that this intuition is correct for this example. So, the answer is $3.5 + 3.5 + 3.5 = 10.5$.

Now try these questions.

8. Suppose I roll a fair **eight** sided die. What is the probability the result is prime?
9. What is the expected value of a single roll of an eight sided die?
10. What is the expected value if I roll a six sided die and an eight sided die, then add the results?

4 Summations and Products

The letter Σ (Greek capital sigma) is used to represent the summation of a (not necessarily finite) sequence of numbers. For example, to represent the sum of the first n positive integers, we would write $\sum_{k=1}^{n} k$ (if we write it inline); sometimes it is written as:

$$\sum_{k=1}^{n} k$$

Similarly, $\sum_{k=1}^{n} k^2$ would be used to indicate the sum of the squares of the first n positive integers.

Some summations have a convenient "closed form" : a formula to represent the full sum that doesn't involve adding the full sequence. You might find the following summations' closed forms to be useful:

- $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
- $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{k=0}^{\infty} x^k, |x| < 1 = \frac{1}{1-x}$
- $\sum_{k=1}^{\infty} kx^{k-1}, |x| < 1 = \frac{1}{(1-x)^2}$

The same concept applies to the letter Π (Greek capital pi) for product. For example, $\prod_{k=1}^{n} k$ is the *product* of the first n positive integers.

You *do not* need to memorize the closed forms – they won't appear on any artifacts for which you do not have your notes. However, you should be able to read a summation or product and know what is being added or multiplied.