CompSci 260P
Spring 2023 Lecture 4: Introducing Complexity

About $\mathcal{NP}$-completeProblems

- In general, $\mathcal{NP}$-completemeans:
  - Can certify a "yes" in polynomial space
  - Can verify that certificate in polynomial time
  - Can use it to solve another problem known to be $\mathcal{NP}$-complete
- Informally, $\mathcal{NP}$-completemeans:
  - We do not know an efficient solution
  - We do not know that one does not exist
- 1972, Cook-Levin:
  (effectively) 3-SAT is $\mathcal{NP}$-complete

3-Sat

$$\phi = (x_2 \vee x_3 \vee x_4)(\overline{x_2} \vee x_3 \vee \overline{x_4})(\overline{x_1} \vee x_3 \vee x_5)$$
$$(\overline{x_1} \vee x_2 \vee x_5)(\overline{x_3} \vee x_4 \vee x_5)(\overline{x_2} \vee \overline{x_4} \vee \overline{x_5})$$
$$(x_1 \vee \overline{x_2} \vee x_5)(x_3 \vee \overline{x_4} \vee x_5)$$

Variable	Truth Value
x_1	True or False
x_2	True or False
x_3	True or False
x_4	True or False
x_5	True or False

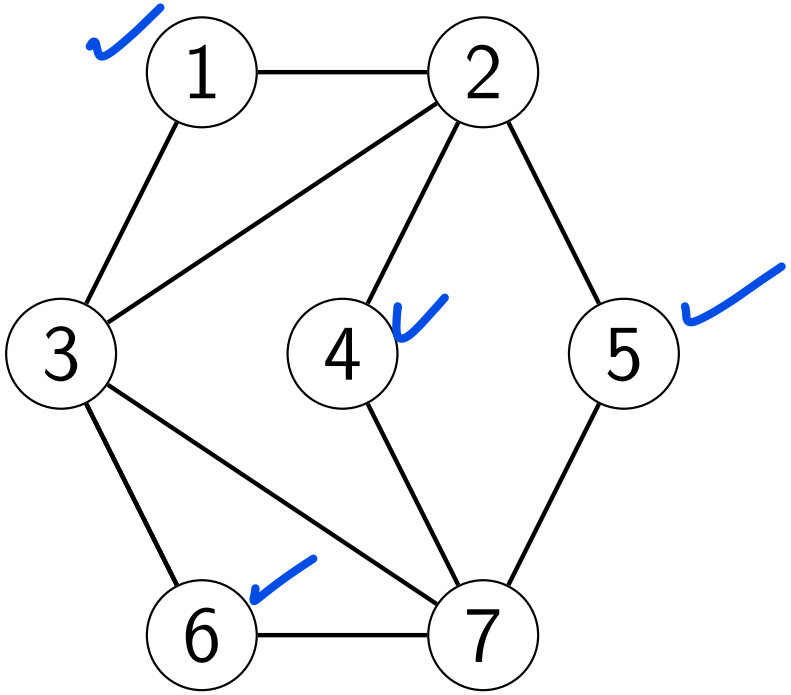
n variables
K clauses

for Ind Set
and 3-color:

$|V| = ?$ $|E| = ?$

Independent Set

Find an independent set of size 4 in this graph:



Verifier for Independent Set

Certificate: V', a set of vertices.
Verifier:

```
if  V' ⊈ V then
    return false
if |V'| ≠ k then
    return false
for all edges e = (u, v) ∈ E do
    if u ∈ V' and v ∈ V' then
        return false
return true
```

Can Independent Set solve 3-Sat?

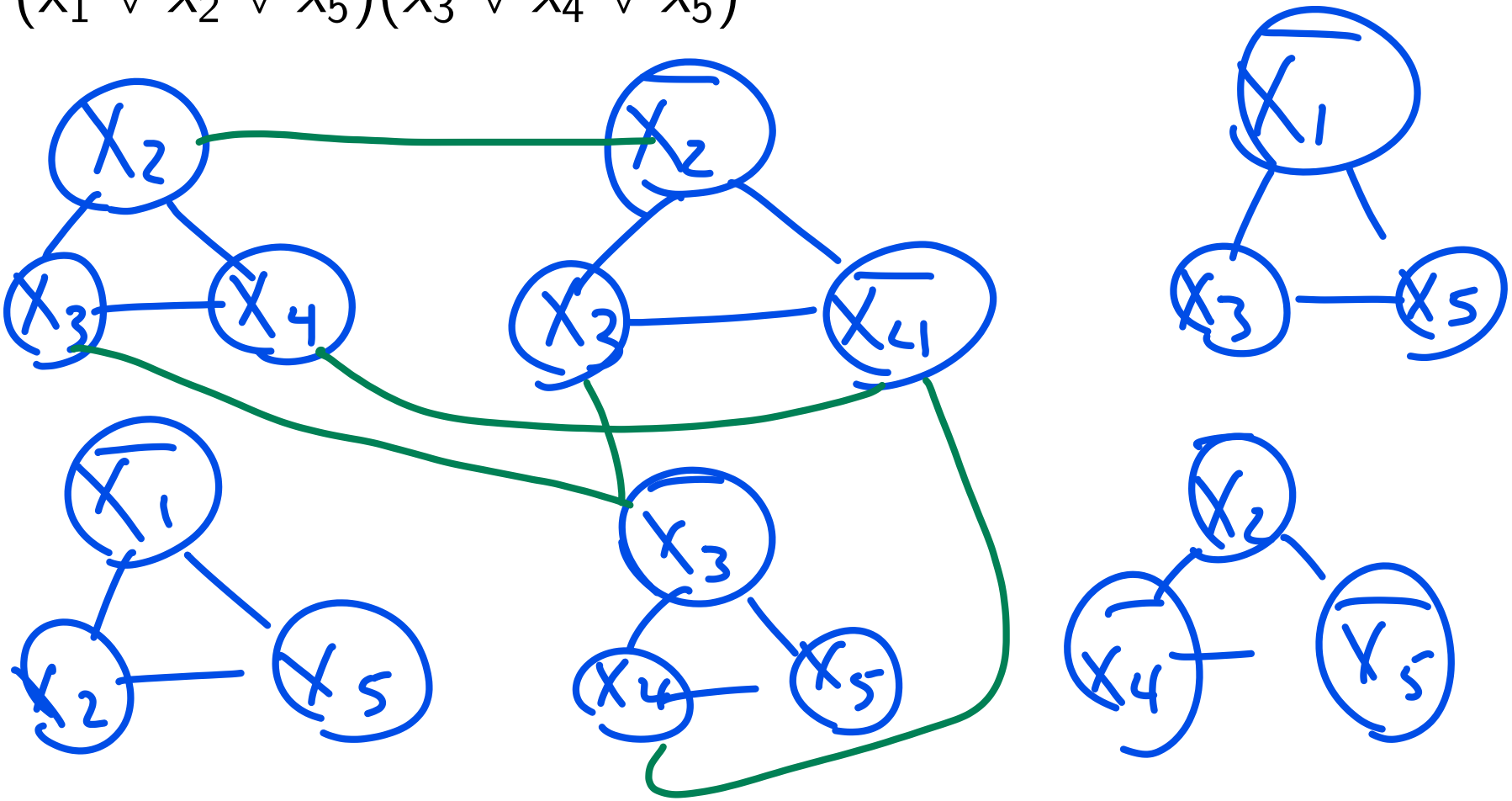
Given a solution to INDEPENDENT SET, use it to write an algorithm that solves 3-SAT.

The Plan: Build a graph such that:

- The INDEPENDENT SET solver will select one variable from each clause, when given the graph and an appropriate value of k as input.
- Before you use the solver, modify the graph so that x_i and $\overline{x_i}$ won't both be selected, for any i.
- It is okay to select x_i two or more times, or $\overline{x_i}$ two or more times, as long as you don't select x_i and $\overline{x_i}$

Reduction

$$\phi = (x_2 \vee x_3 \vee x_4)(\overline{x_2} \vee x_3 \vee \overline{x_4})(\overline{x_1} \vee x_3 \vee x_5)$$
$$(\overline{x_1} \vee x_2 \vee x_5)(\overline{x_3} \vee x_4 \vee x_5)(\overline{x_2} \vee \overline{x_4} \vee \overline{x_5})$$
$$(x_1 \vee \overline{x_2} \vee x_5)(x_3 \vee \overline{x_4} \vee x_5)$$



Reduction

3-SAT(n variables, k clauses)

for each clause $A \vee B \vee C$ **do**

Create 3 vertices // one each A, B, C

Add edges $(A, B), (B, C), (A, C)$

for each variable x_i **do**

Connect any "x_i" node to all $\overline{x_i}$ nodes

return INDEPENDENT SET(G, k)

Could we get false positives?

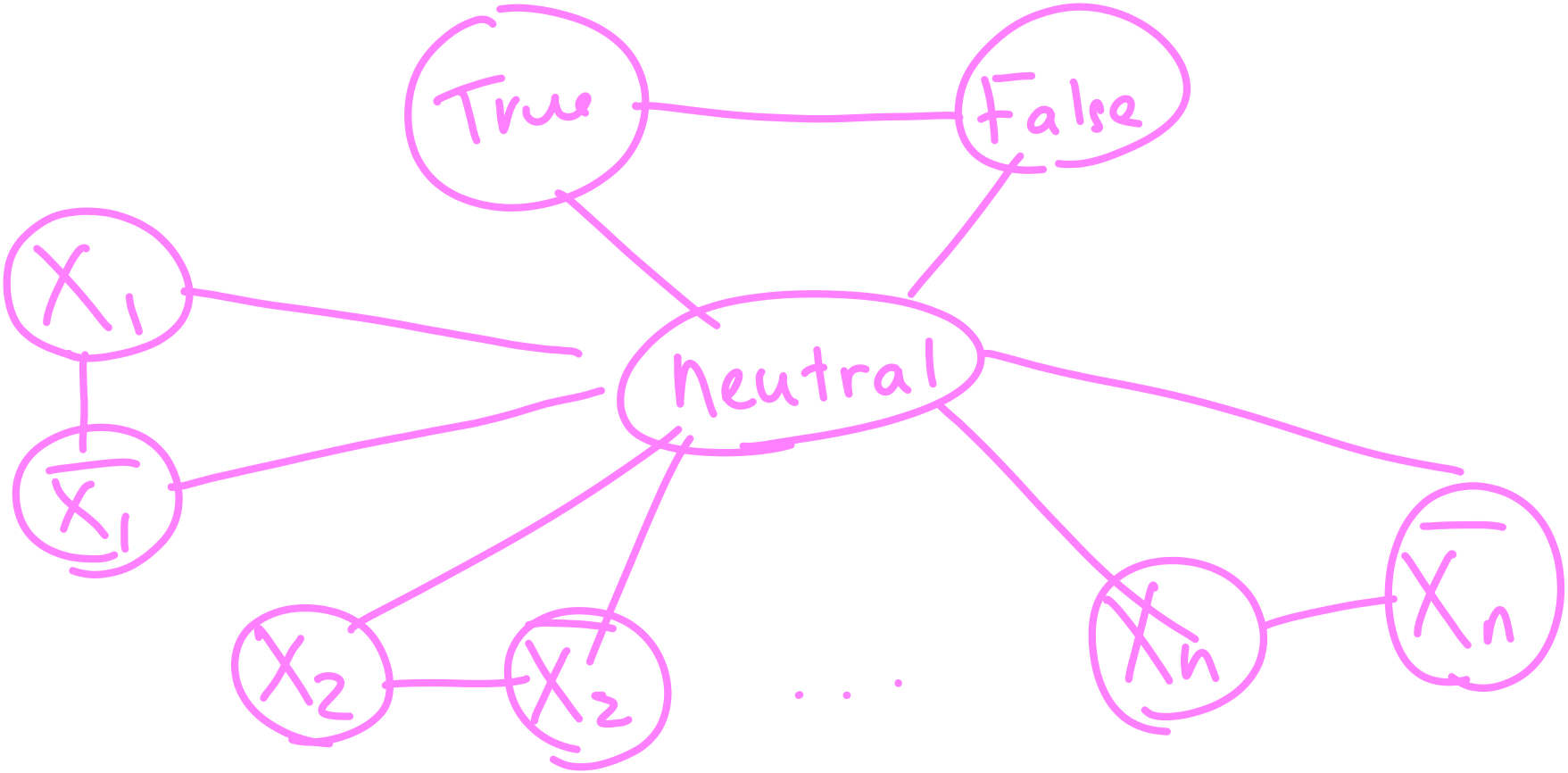
Show that, if that graph G has an independent set of size k, then the 3-$\textsc{Sat}$ instance truly has a satifying assignment.

Could we get false negatives?

Show that, if the 3-SAT instance we have as input has a satisfying assignment, the graph we build will have an independent set of size k.

Use 3-COLOR to get a truth value assignment on n variables

- Remember, all 2^n TVAs should be possible.
- Running time polynomial plus call to 3-COLOR



Amend the 3-COLOR usage to get a satisfying truth value assignment on n variables.

For each clause $A \vee B \vee C$, add as follows:

e.g. if $(\overline{x_2} \vee x_3 \vee \overline{x_4})$, then $A = \overline{x_2}$ etc