CompSci 162
Spring 2023 Lecture 21:
Independent Set and Vertex Cover are $\mathcal{NP}$-complete

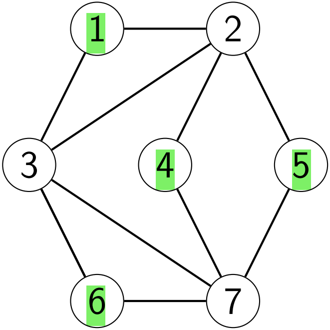
Brief Review

- $\mathcal{NP}$: (decision) problems that can be verified efficiently (whose "yes" instances)
- SAT $= \{\langle\phi\rangle : \phi$ is a satisfiable Boolean formula.$\}$
  Ex: $(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6})$
- Every SAT can be one with 3 literals per clause
- SAT and 3-SAT are $\mathcal{NP}$-complete
  - A solver exists for every $X \in \mathcal{NP}$:
    1. Set up X as instance of SAT
    2. Instance size is polynomial function of X
    3. Solve that SAT instance $=$ solve X

poly time solution for SAT
= poly time solver for ALL $X \in NP$

Independent Set

Find an Independent Set of size $k = 4$ in this graph:



Independent Set is $\mathcal{NP}$-complete

- Always prove $\in \mathcal{NP}$ first.
- Then use Independent Set to solve an $\mathcal{NP}$-complete problem
  - Right now, 3-Sat
  - In general, any known one is fine
  - For this class, use one shown in class (lecture or discussion)
- Solver must be polynomial (not including call to Independent Set)

Independent Set is in $\mathcal{NP}$

1. - NTM polynomial-time decider ←
   - DTM polynomial-time verifier ←

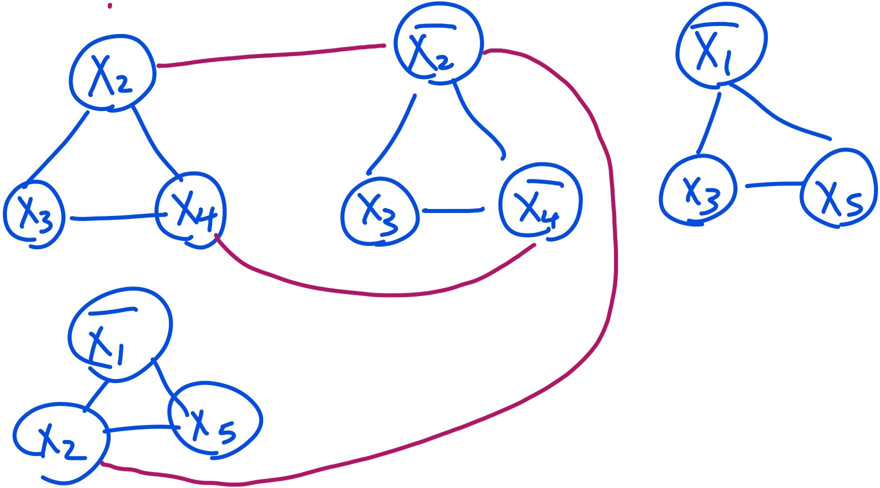
1. Non-det select K vertices
2. verify each is vertex in G
3. verify K distinct chosen
4. for each e = (u,v) in G
   verify !(u,v both chosen)

1. Certificate:
   K vertices

3-SAT Solver via Independent Set

Input: 3-Sat instance ϕ

Ex: $(x_2 \vee x_3 \vee x_4)(\overline{x_2} \vee x_3 \vee \overline{x_4})(\overline{x_1} \vee x_3 \vee x_5)(\overline{x_1} \vee x_2 \vee x_5)\ldots$



Reduction

3-Sat(n variables, k clauses)

for each clause $A \vee B \vee C$ **do**

Create 3 vertices // one each A, B, C

Add edges $(A, B), (B, C), (A, C)$

for each variable x_i **do**

Connect any "x_i" node to all $\overline{x_i}$ nodes

return Independent Set(G, k)

Could we get false positives?

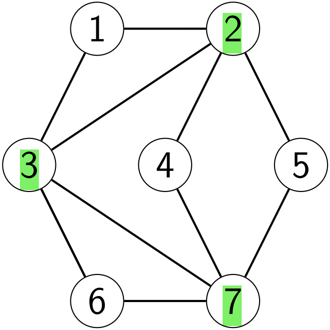
Show that, if that graph G has an independent set of size k, then the $3\text{-}\textsc{Sat}$ instance truly has a satifying assignment.

Could we get false negatives?

Show that, if the 3-SAT instance we have as input has a satisfying assignment, the graph we build will have an independent set of size k.

Vertex Cover

Find a Vertex Cover of size $k = 3$ in this graph:



Prove Vertex Cover is $\mathcal{NP}$-complete

- Always prove $\in \mathcal{NP}$ first. do as reinforcement.
- Then use Vertex Cover to solve an $\mathcal{NP}$-complete problem

Indep-Set (G, K)
{

Vertex-Cover (G, |G.V| - K)

}

Correctness of Reduction

Claim: A graph G has an independent set of size $n - k$ if and only if it has a Vertex Cover of size k

do as reinforcement

Set Cover

Set Number	Elements
1	A B
2	A C D E
3	B C F I
4	D G
5	E H
6	I J
7	F G H J

Will start lecture 5/31 with this.

Try to prove this is NP-complete before class.

Idea For Reduction