CompSci 162
Spring 2023 Lecture 1 and 2
Sets and Functions

Let's see some functions

Given sets A and B,

- What does $|A| = |B|$ mean?

  - Cardinality same
  - equinumerable
  - $\exists$ bijective f $A \longleftrightarrow B$

    f(guacamole) = Colossus of Rhodes

$|A| \leq |B|$? injective, possibly bijective

$|A| < |B|$? injective not bijective

How many naturals?

in 162, $0 \in N$

- $\mathbb{N} = \{0, 1, 2, \ldots\}$
- $|\mathbb{N}| = \aleph_0$ "countably infinite"
- How many odd numbers are there?

$f(x) = 2x+1$ ~~injective~~

$x \in N$ odd bijective

injective: $f(i) = f(j) \Rightarrow i \overset{?}{=} j$

surjective: $\frac{x-1}{2} \in N$

How about $\mathbb{Z}$?

$$\mathbb{Z} = \{ \ldots -2, -1, 0, 1, 2 \ldots \}$$

$$\bar{0} \quad \bar{1} \quad \bar{2} \ldots$$

$$6 \quad 4 \quad 2 \quad 0 \quad 1 \quad 3 \quad 5$$

$$\left.\begin{aligned} f(2K) &= -K \\ f(2K+1) &= K+1 \end{aligned}\right\} \text{one function in two pieces}$$

$\exists$ bijective

"one piece" function

$$f(n) = \frac{1 + (-1)^n(2n+1)}{4}$$

How about $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$?

$P_i = \{ (x, y, z) \mid x \in \mathbb{N},\ y \in \mathbb{N},\ z \in \mathbb{N} \text{ and } x+y+z == i \}$

Sort order: by i, then within P_i
Sort within P_i by x (y breaks ties)

$i = x+y+z$

Within P_i, where is (x, y, z) in order?

$= (\# x' < x) + (\# x' = x \text{ and } y' < y)$

$$\sum_{j=0}^{x-1} (i+1-j) + y \rightsquigarrow \frac{x(2i - x + 3)}{2} + y$$

How about $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$? (continued)

We saw that (x, y, z) is the $\frac{x(2\cancel{n}\,i - x + 3)}{2} + y$ th element in $P_{\cancel{n}\,i}$, where $\cancel{n}\,i = x + y + z$.

But how many total elements are in $P_0, P_1, \ldots P_{\cancel{n}\,i-1}$?

$= \#$ ways $x+y+z \leq i-1$

We need to add the above value to that count

$i-1$ pennies 4 jars

C = place penny : $i-1$

N = next jar : 3

$$\binom{i-1+3}{3} = \binom{i-1+3}{i-1} = \binom{i+2}{3} = \frac{(i+2)(i+1)i}{6}$$

$$f(x,y,z) = \frac{x(2i-x+3)}{2} + y + \frac{(i+2)(i+1)i}{6}$$

How about $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$?

Take two, but this time, with Schröder-Bernstein Theorem

$g : \mathbb{N} \to \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ injective

$g(n) = (n, n, n)$

$f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ injective

$f(x, y, z) = 2^x 3^y 5^z$

$\mathbb{Q}$ is countable

$q \in \mathbb{Q}$ means $\exists k, m$ such that:

- $k, m \in \mathbb{Z}$
- $k \geq 0$
- k, m are relatively prime
- $q = k/m$

$f : \mathbb{Q} \to \mathbb{N}$

$$f(k, m) = \begin{cases} 2^k 3^m & m > 0 \\ 2^k 5^{-m} & m < 0 \end{cases}$$

$g(n) = n/1$