# CompSci 162
# Spring 2023 Lecture 14:
# Closures of Context-Free Languages

# CFLs: Closed Under Union

If $L_1$ and $L_2$ are CFLs, then so is $L_3 = L_1 \cup L_2$

- $G_1 = (V_1, \Sigma_1, R_1, S_1)$ for $L_1$
- $G_2 = (V_2, \Sigma_2, R_2, S_2)$ for $L_2$

$S_3 \to S_1 \mid S_2$

$G_3 = (V_1 \cup V_2 \cup \{S_3\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{S_3 \to S_1, S_3 \to S_2\}, S_3)$

# CFLs: Closed Under Concatenation

If $L_1$ and $L_2$ are CFLs, then so is $L_4 = L_1 L_2$.
Proof looks a lot like Union:

- $G_1 = (V_1, \Sigma_1, R_1, S_1)$
- $G_2 = (V_2, \Sigma_2, R_2, S_2)$
- $G_4 = (V_1 \cup V_2 \cup \{S_4\}, \Sigma_1 \cup \Sigma_2, R_1 \cup R_2 \cup \{\}, S_4)$

$S_4 \rightarrow S_1 S_2$

# CFLs: Closed Under Kleene Star

If $L_1$ is a CFL, then so is $L_5 = L_1^*$.
Proof looks a lot like Union:

- $G_1 = (V_1, \Sigma_1, R_1, S_1)$
- $G_5 = (V_1 \cancel{\cup V_2} \cup \{S_5\}, \Sigma_1 \cancel{\cup \Sigma_2}, R_1 \overset{\cup}{\cancel{\cup R_2 \cup}} \{\}, S_5)$

$S_5 \to S_5 S_5 \mid S_1 \mid \varepsilon$

## Intersection of CFLs and Regular Languages

Sheep → baa
Sheep → baa Sheep

The intersection of a CFL and a Regular Language

- PDA $M_1 = (K_1, \Sigma, \Gamma_1, \Delta_1, s_1, F_1)$
- DFA $M_2 = (K_2, \Sigma, \delta, s_2, F_2)$
- Result PDA is $M = (K, \Sigma, \Gamma, \Delta, s, F)$ where:
  - States: $K =$ $K_1 \times K_2$ (all pairs)
  - Stack alph: $\Gamma =$ $\Gamma_1$
  - start: $s =$ $(s_1, s_2)$
  - accepts: $F =$ $F_1 \times F_2$
  - transition: What is $\Delta$?

$q_1 \xrightarrow{a,b \to g} p_1$

for all non-$\varepsilon$ tr. in $M_1$

$(q_1, q_2) \xrightarrow{a,b \to g} (p_1, \delta(q_2, a))$

for all $\varepsilon$-tr. in $M_1$

$q_1 \xrightarrow{\varepsilon, b \to g} p_1$ in $M_1$

$(q_1, q_2) \xrightarrow{\varepsilon, b \to g} (p_1, q_2)$

# Are CFLs closed under intersection?

If $L_1$ and $L_2$ are CFLs, is $L_7 = L_1 \cap L_2$ CFL? No.

Suppose FSOC that **yes**.

$$L_1 = \{ a^n b^n c^m : m, n \in \mathbb{N} \}$$

$$L_2 = \{ a^m b^n c^n : m, n \in \mathbb{N} \}$$

$$L_1 \cap L_2 = \{ a^n b^n c^n : \cancel{m}, n \in \mathbb{N} \},$$

but that is not CF $\rightarrow\leftarrow$

# Are CFLs closed under complement?

If $L_1$ is a CFL, is $L_8 = \overline{L_1}$?

$$\overline{L_1} \text{ and } \overline{L_2} \quad \text{if yes}$$

$$\overline{\overline{L_1} \cup \overline{L_2}} = L_1 \cap L_2$$

So no. $\rightarrow\leftarrow$