Intro to Undecidability

Question 1. Is $A_{TM} = \{\langle M, w \rangle : M$ is a Turing Machine and M accepts $w\}$ a Turing-recognizable language?

Question 2. How many Turing Machines exist? I don't mean how many were built in the physical world, but rather how many possible Turing Machines are there?

Question 3. How large is the set of all infinite binary sequences?

Question 4. How large is the set of all languages? For context of what I am asking, recall that the set of strings accepted by a given automaton (including Turing Machines) is a *language*.

Question 5. Is $A_{TM} = \{\langle M, w \rangle : M$ is a Turing Machine and M accepts $w\}$ a Turing-**decidable** language?

We say a language is **co-Turing-recognizable** if it is the complement of a Turing-recognizable language.

Question 6. What can we say if a language is both Turing-recognizable and also co-Turing-recognizable? Prove that this is the case.

Question 7. We saw that $A_{TM} = \{\langle M, w\rangle : M$ is a Turing Machine and M accepts $w\}$ a Turing-**recognizable** language. What about $\overline{A_{TM}}$?

The Halting Problem

Question 8. The *Halting Problem* is as follows. The language $HALT_{TM} = \{\langle M, w\rangle : M$ is a Turing Machine and M halts on input $w\}$. Show that $HALT_{TM}$ is undecidable. Why is this easier than the undecidable proofs we did last time?

Question 9. Show that $E_{TM} = \{\langle M \rangle : M \text{ is a Turing Machine and } L(M) = \emptyset\}$ is undecidable.

Suppose FSOC that E_{TM} is decidable and I want to decide A_{TM} given input $\langle M, w \rangle$.

Question 10. Show that $EQ_{TM} = \{\langle M_1, M_2 \rangle : M_1 \text{ and } M_2 \text{ are Turing Machines and } L(M_1) = L(M_2)\}$ is undecidable.

Hint: is there any language one of those machines could recognize where the equality of it and a mystery machine would be useful information for you?

General Strategy to prove X is undecidable

- Suppose I had a TM that decides X
- Pick an undecidable problem Y
- Write a TM to decide Y
  - Must be a valid TM **EXCEPT**
    it assumes existence of TM to decide X
- But that would decide Y
- By contradiction, no TM for X