CompSci 161
Winter 2023 Lecture 05:
Divide and Conquer II:
QuickSort and Order Statistics

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QuickSort Step 1: Partition

85	24	63	45	17	31	96	50

1. Choose a pivot.
2. Place that pivot x in the right spot.
3. Pivot the rest of the array.

$$\boxed{\quad < x \quad \mid x \mid \quad > x \quad}$$

3

QuickSort

85	24	63	45	17	31	96	50
17	24	31	45	50	63	96	~~17~~ 85

4

How fast is QuickSort?

- $T(n) = T(\text{lower}) + T(\text{upper}) + \Theta(n)$
- If lower and upper are both size $n/2$?

$$T(n) = 2T(n/2) + n = \theta(n \lg n)$$

- What if we select a pivot uniformly at random?
- What if we could find a median in $\Theta(n)$...

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Average Case Analysis of QuickSort

Suppose

- All permutations equally likely
- All n values are distinct (for simplicity)
- Define $S_1, S_2, \ldots S_n$ as sorted order.

Let $P_{i,j}$ be probability we compare S_i and S_j.

$$P_{i,j} = \frac{\#\text{yes}}{\#\text{total}} = \frac{2}{j-i+1}$$

6
Expected number of comparisons

$$
\begin{aligned}
E\Big(\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_{i,j}\Big) &= \sum_{i=1}^{n}\sum_{j=i+1}^{n} E(X_{i,j}) \\
&= \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{2}{j-i+1} \\
&= \sum_{i=1}^{n}\sum_{k=2}^{n-i+1} \frac{2}{k} \\
&< \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{2}{k}
\end{aligned}
$$

$X_{i,j}$: IRV

harmonic

is $\Theta(n \lg n)$

7 # The Selection Problem

- Given a list S and numeric k
- Want: if we sorted S, what is S_k?
- Brute force:
  - Sort S in $\Theta(n \log n)$
  - Return S_k
- Can we do better?

8 # Randomized Selection

quickSelect(S, k)

If n is small, brute force and return.

Pick a random $x \in S$ and put rest into: aka partition

L, elements smaller than x

G, elements greater than x

if $k \leq |L|$ **then**

return quickSelect(L, k)

else if $k == |L| + 1$ **then**

return x

else

return quickSelect(G, k-|L|-1)

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Randomized Selection

- What is the worst-case running time?
- What would cause that bad time?
- Estimate the *expected* running time?
  Hint: on average, the pivot is the median.

$$T(n) = T(n/2) + n$$

is $\Theta(n)$