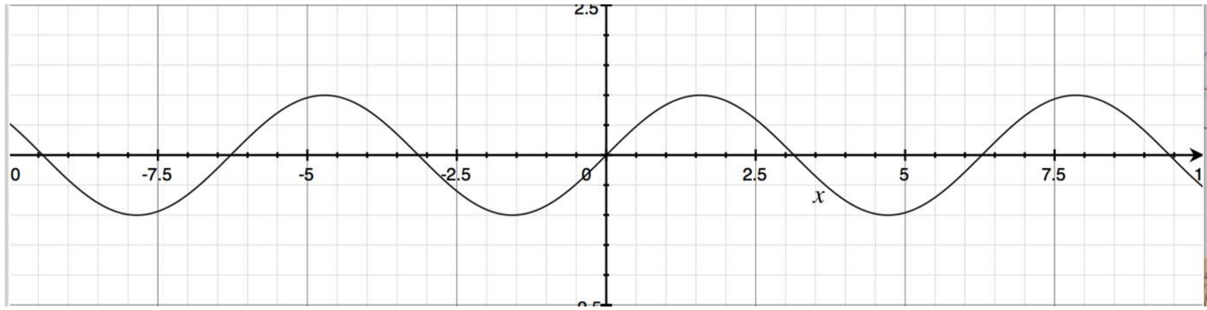
Fourier Transform

Fourier Transform

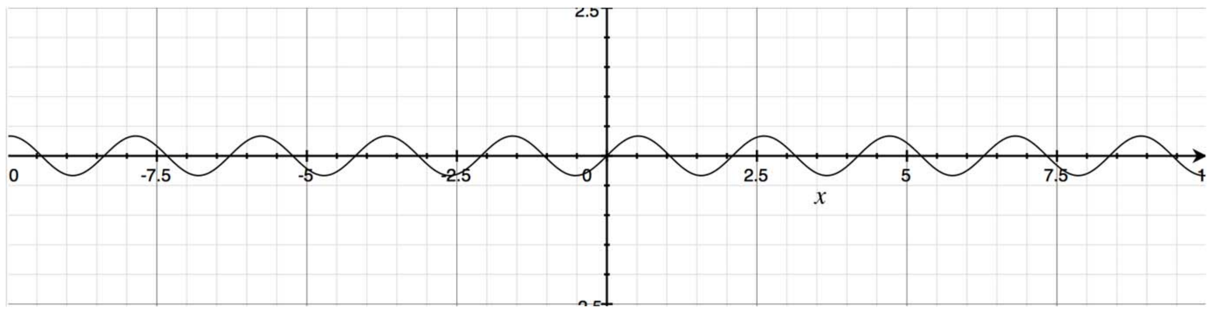
- Any signal can be expressed as a linear combination of a bunch of sine gratings of different *frequency*
  - Amplitude
  - Phase

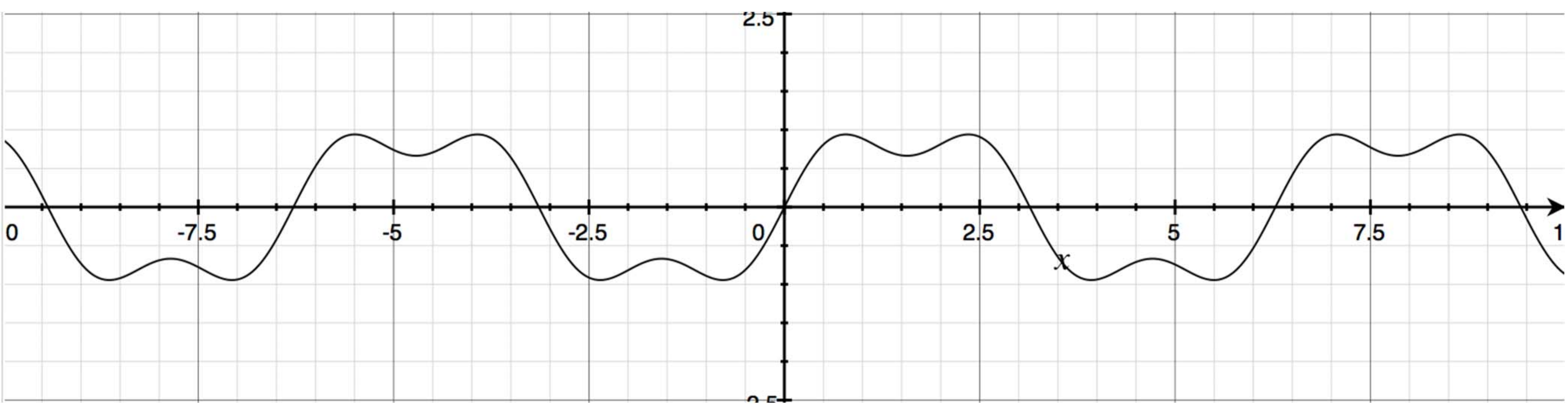
$Sin(x)$



+

$\frac{1}{3}Sin(3x)$



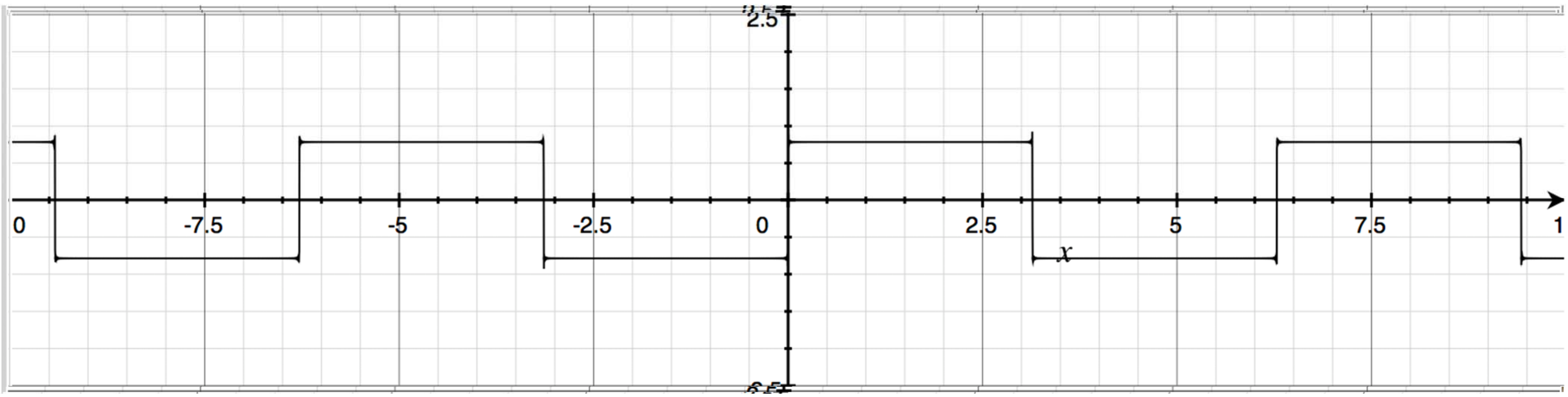


$$Sin(x) + \frac{1}{3}\sin(3x)$$

$$Sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x)$$

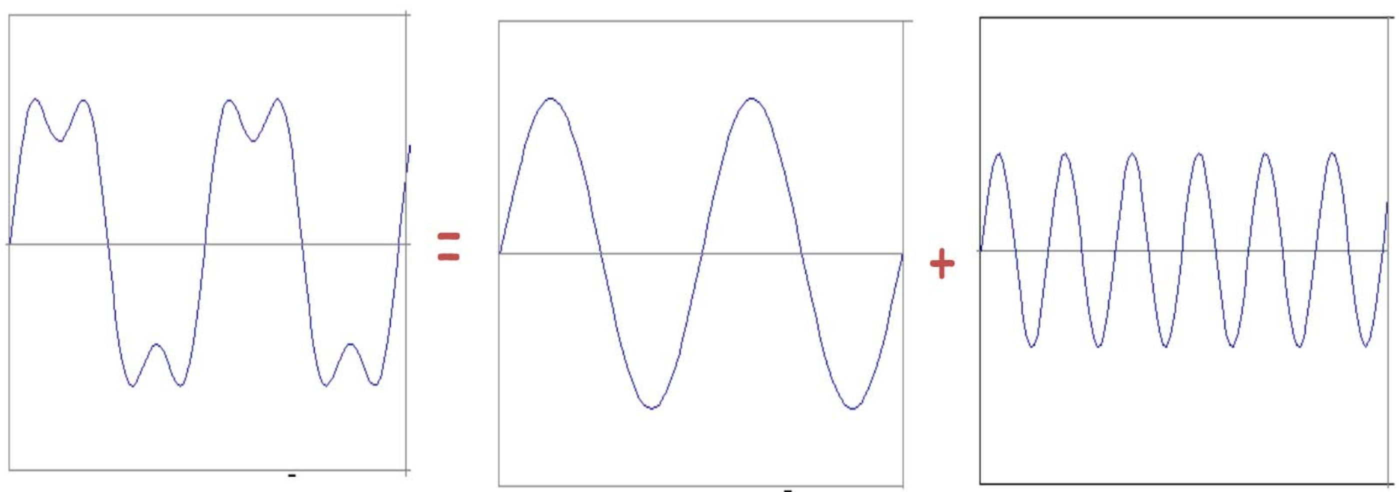
$$Sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)$$

$$Sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x) + \cdots$$



Fourier Transform

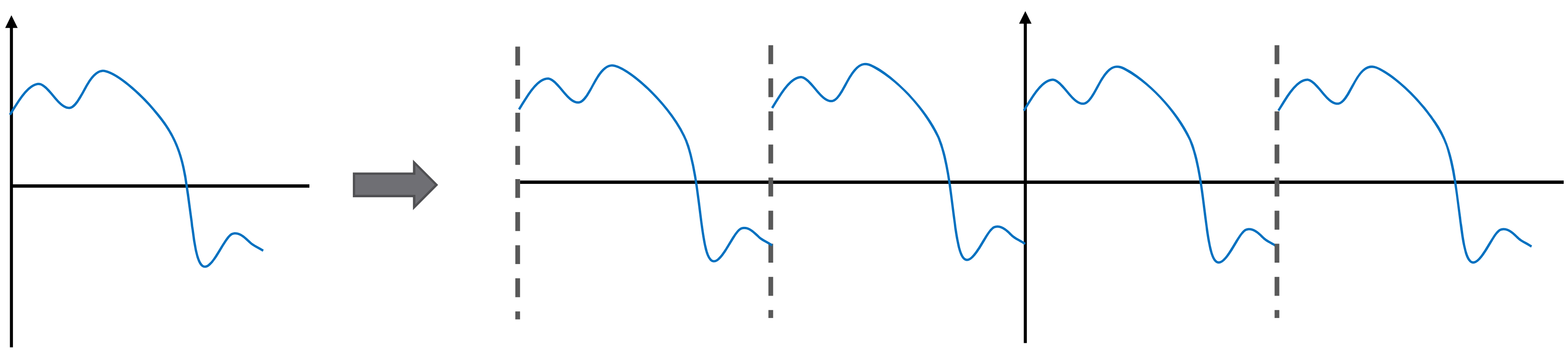
- *Input:* infinite periodic signal
- *Output:* set of sine and cosine waves which together provide the input signal



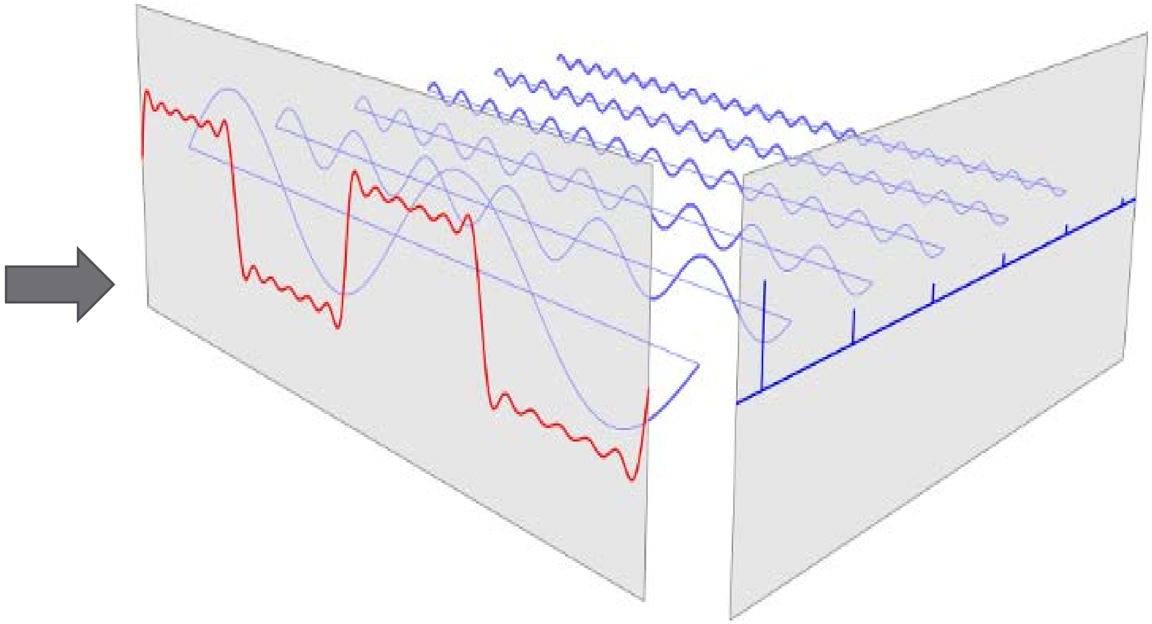
$$g(t) = \sin(2\pi f\, t) + (1/3)\sin(2\pi(3f)\, t)$$

Fourier Transform

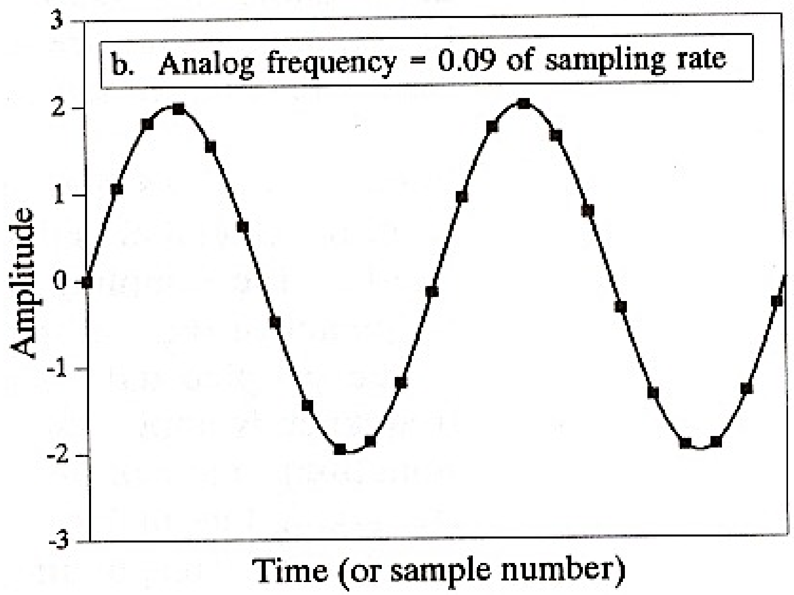
- Digital Signals
  - Hardly periodic
  - Never infinite



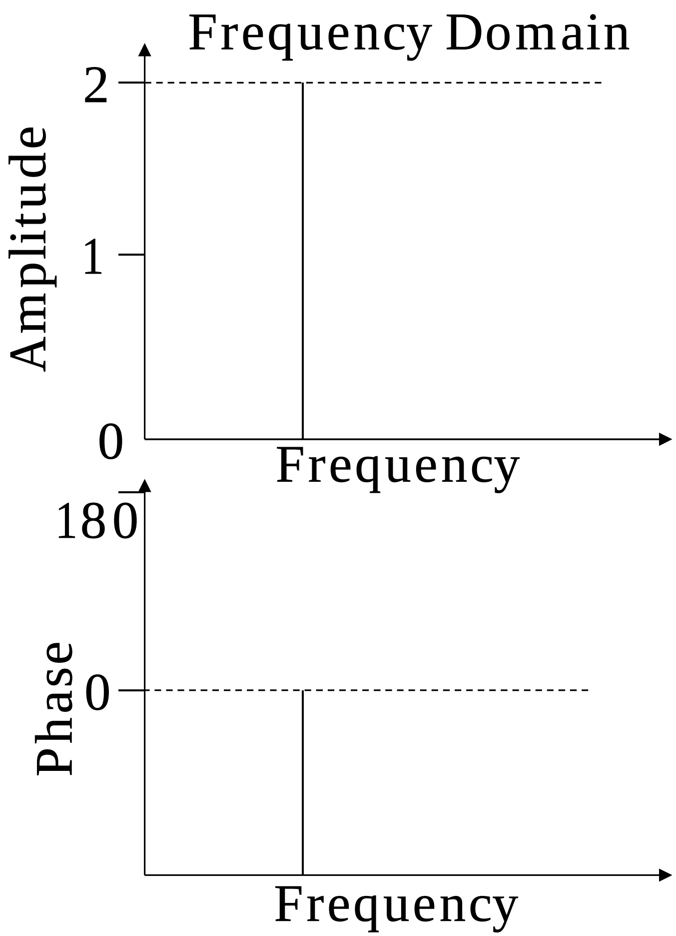
Fourier Transform in 1D



Representation in Both Domains



Time Domain



Discrete Fourier Transform

- DFT decomposes x into $\frac{N}{2} + 1$ cosine and sine waves
- Each of a different frequency

$$x[i] = \sum_{k=0}^{\frac{N}{2}} x_c[k] cos\left(\frac{2\pi ki}{N}\right) + \sum_{k=0}^{\frac{N}{2}} x_s[k] sin\left(\frac{2\pi ki}{N}\right)$$

Time/Spatial Domain

x[0, 1, … N-1]

Inverse DFT/ Synthesis

DFT / Analysis

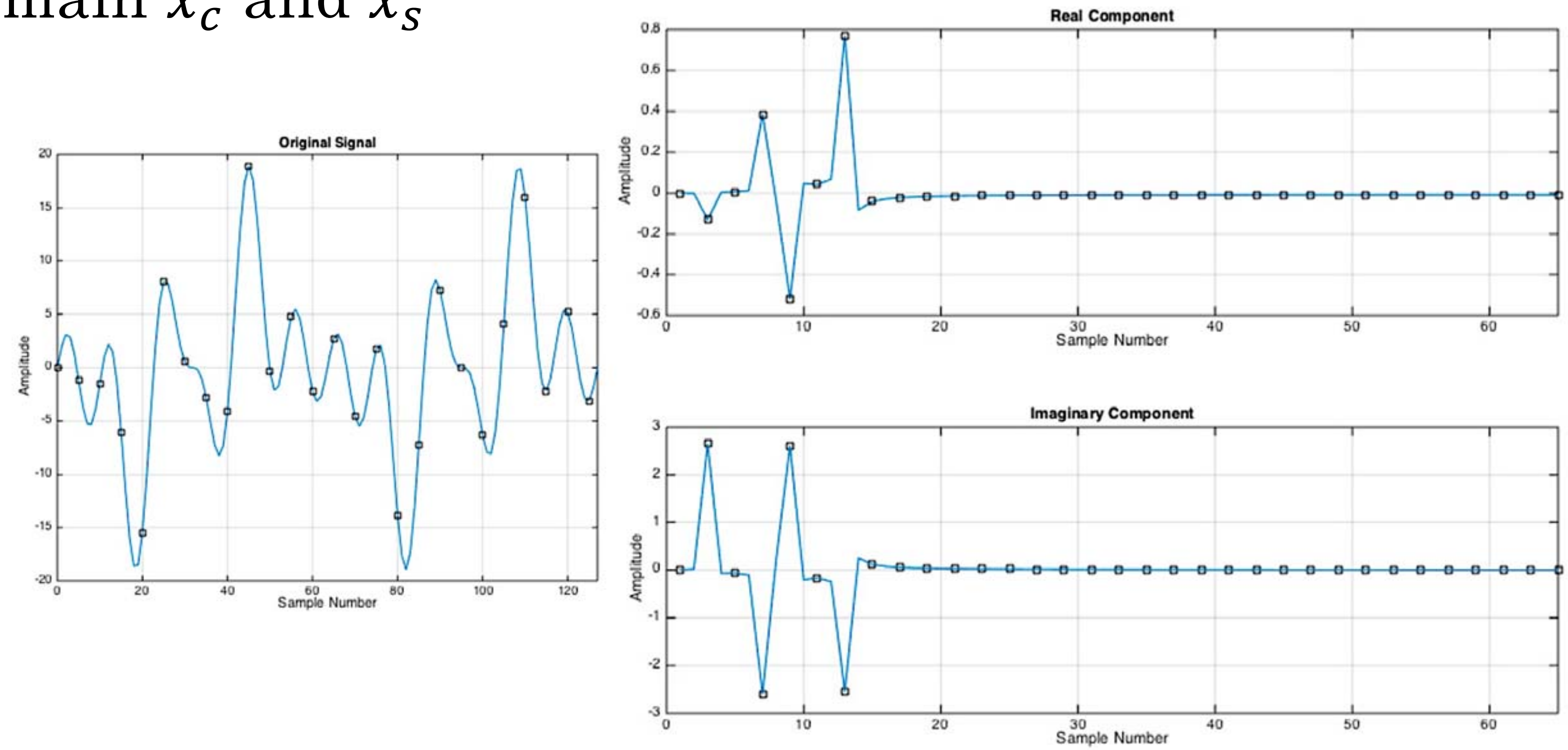
Frequency Domain

x_r[0, 1, … N/2]

x_i[0, 1, … N/2]

DFT - Rectangular Representation

- Decomposition of the time domain signal x to the frequency domain x_c and x_s



Polar Notation

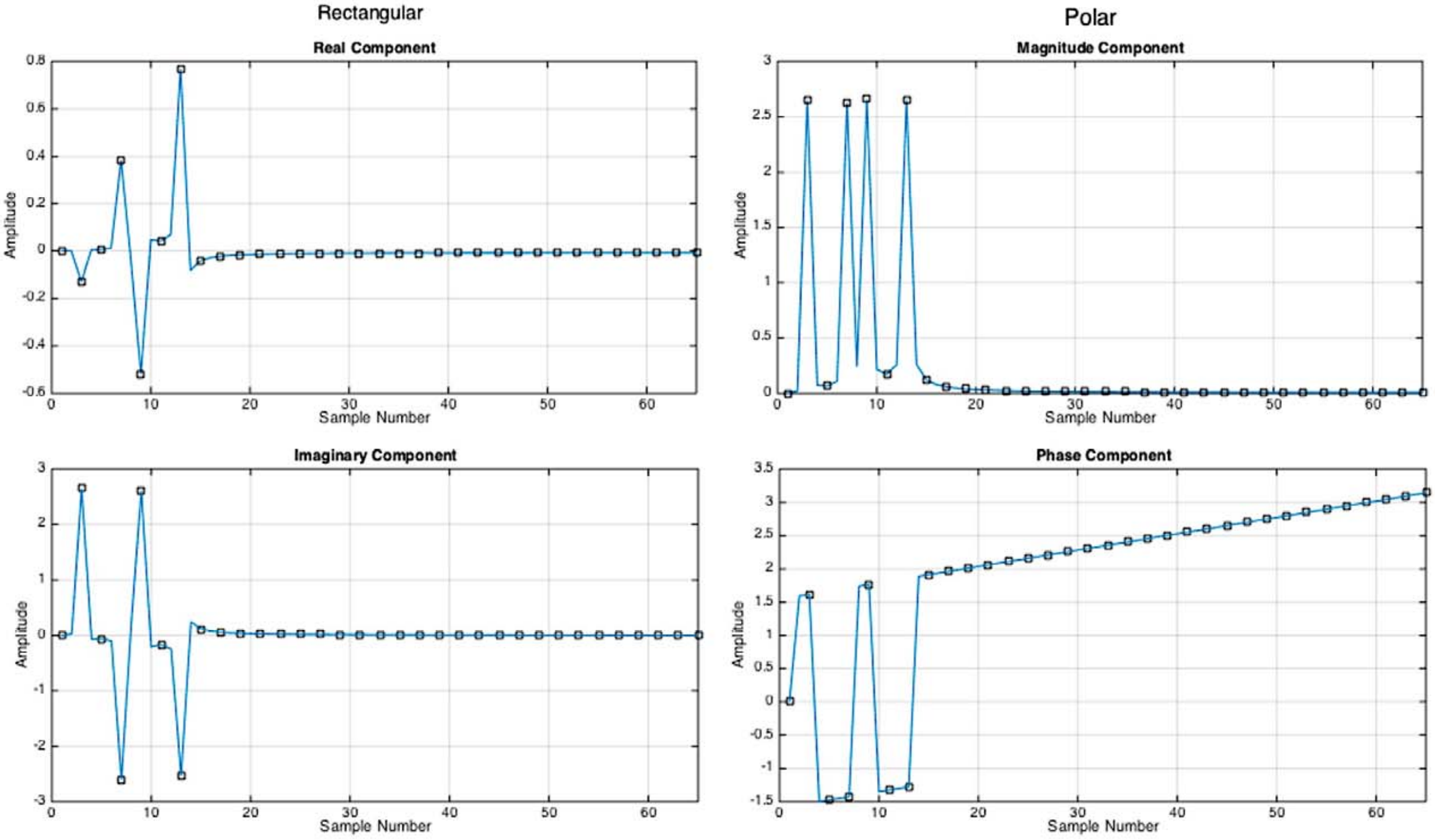
- Sine and cosine waves are phase shifted versions of each other

$$x_c[k]cos(\omega i) + x_s[k]sin(\omega i) = M_k cos(\omega i + \theta_k)$$

$$M_k = \sqrt{x_c[k]^2 + x_s[k]^2}$$ **Amplitude**

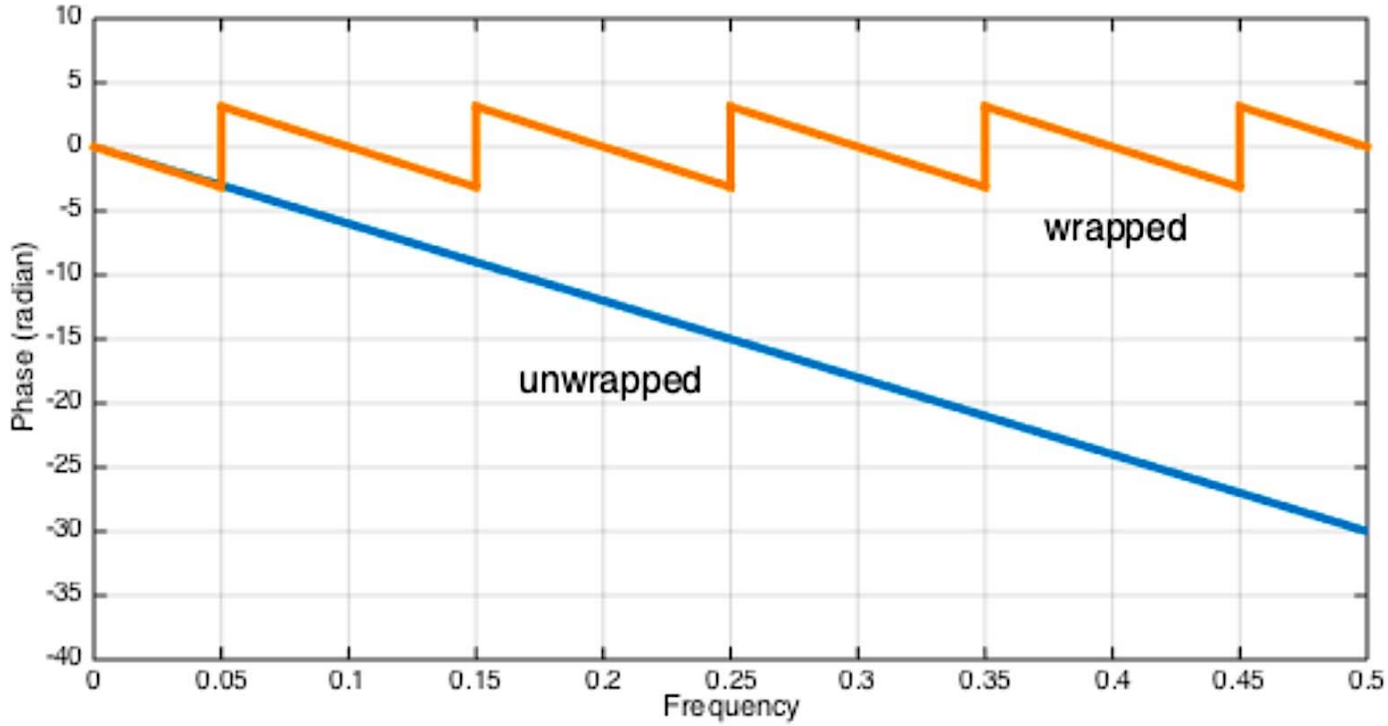
$$\theta_i = tan^{-1}\left(\frac{x_s[k]}{x_c[k]}\right)$$ **Phase**

Polar Representation



Polar Representation

- Unwrapping of phase



Properties

- Homogeneity

$$x[t] \rightarrow (M[f], \theta[f]) \implies kx[i] \rightarrow (kM[f], \theta[f])$$

- Additivity

$$x[t] \rightarrow (x_c[f], x_s[f]), y[t] \rightarrow (y_r[f], y_i[f])$$
$$\implies x[t] + y[t] \rightarrow (x_c[f] + y_r[f], x_s[f] + y_i[f])$$

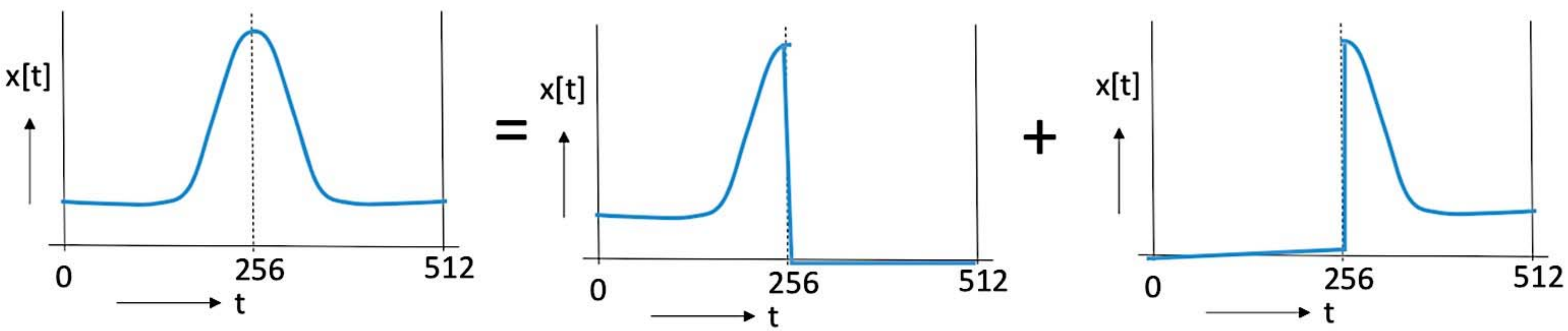
Properties

- Linear phase shift

$$x[t] \rightarrow (M[f], \theta[f]) \implies x[t+s] \rightarrow (M[f], \theta[f] + 2\pi f s)$$

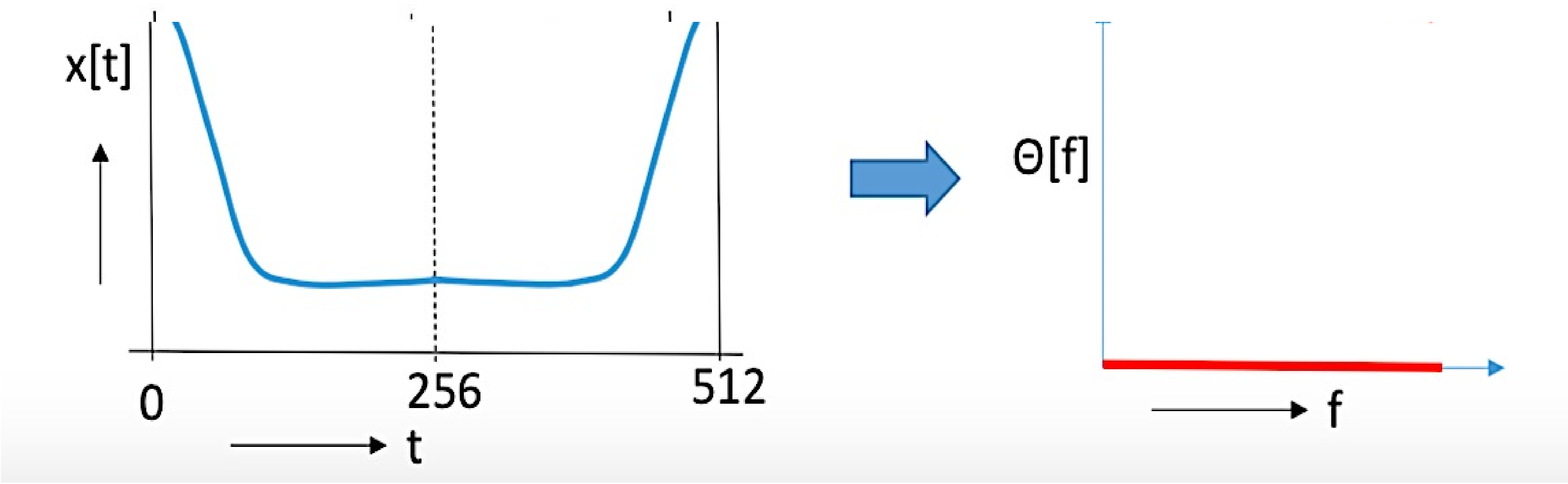
Symmetric Signals

- Symmetric signal always has zero phase

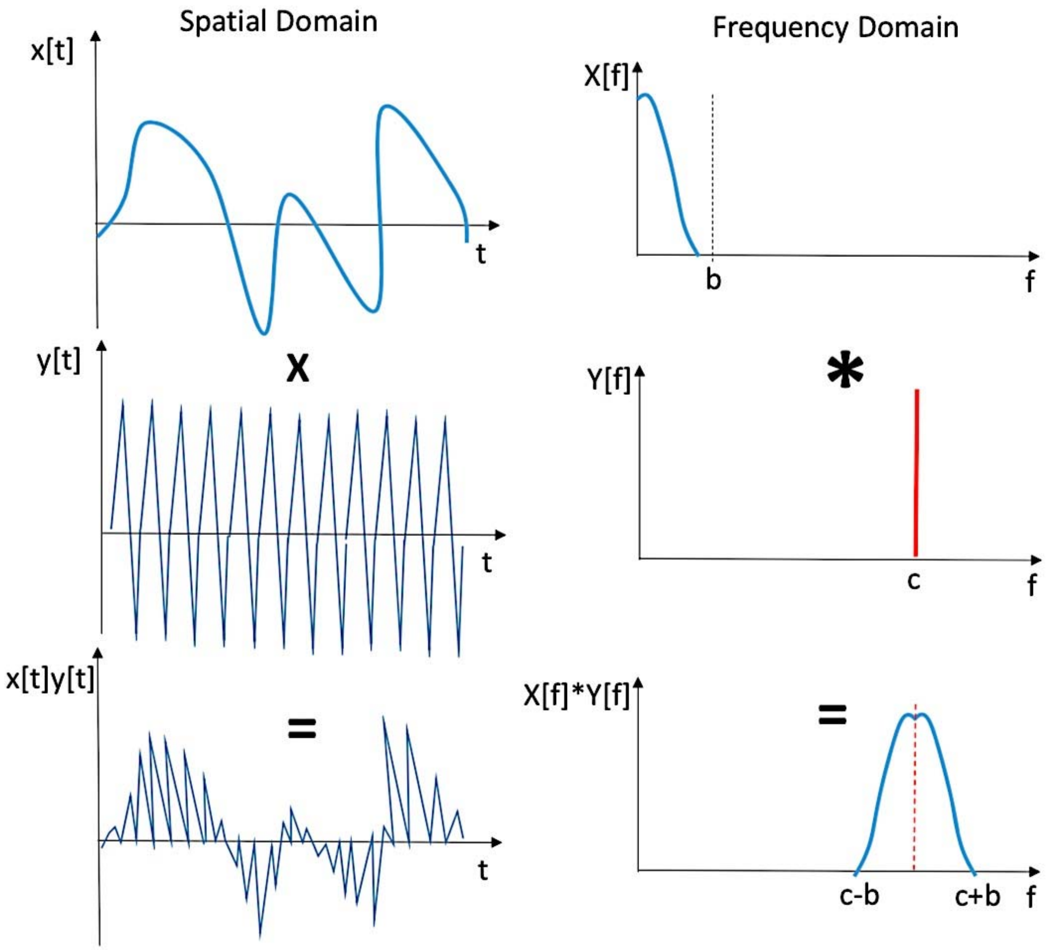


Symmetric Signals

- Frequency response and circular movement



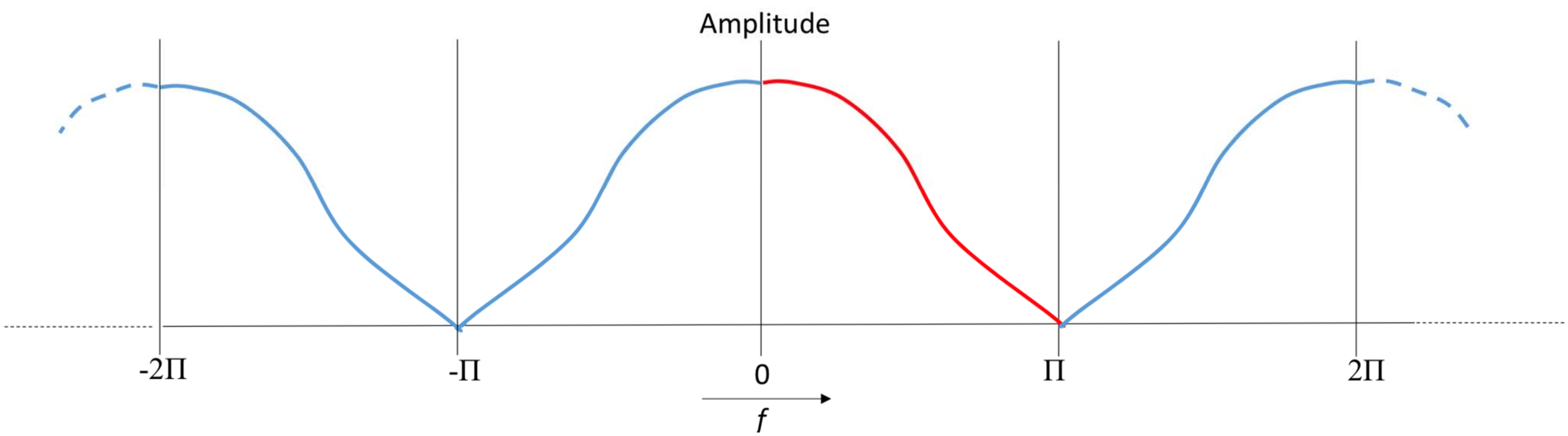
Amplitude Modulation



Periodicity of Frequency Domain

- Amplitude Plot

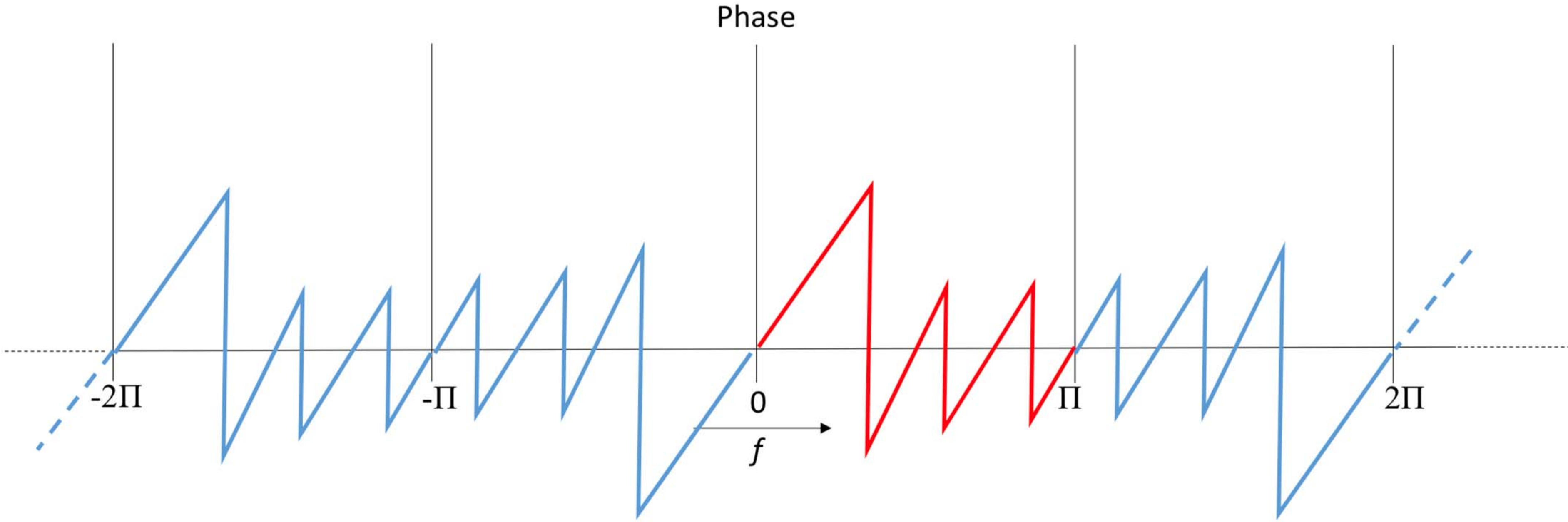
$$M[f] = M[-f]$$



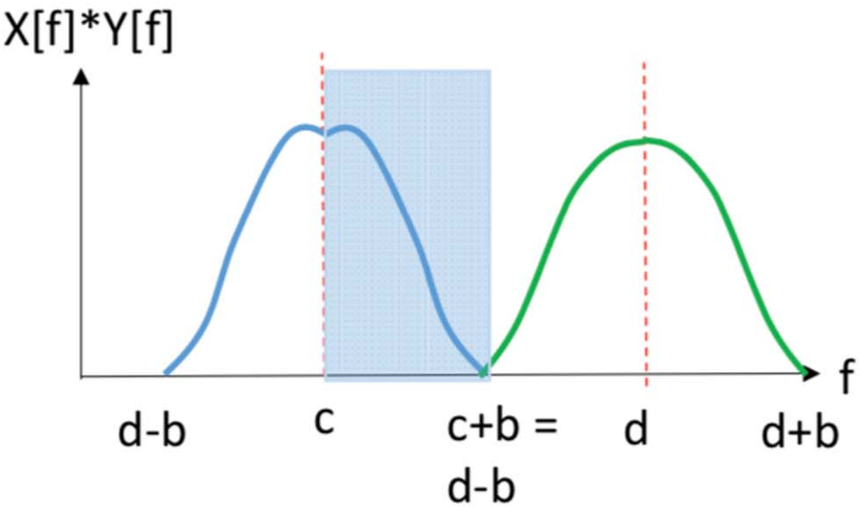
Periodicity of Frequency Domain

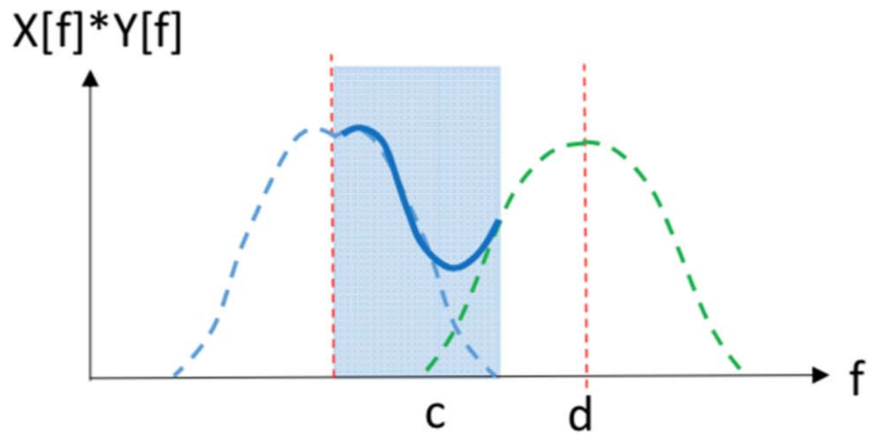
- Phase Plot

$$\theta[f] = -\theta[-f]$$

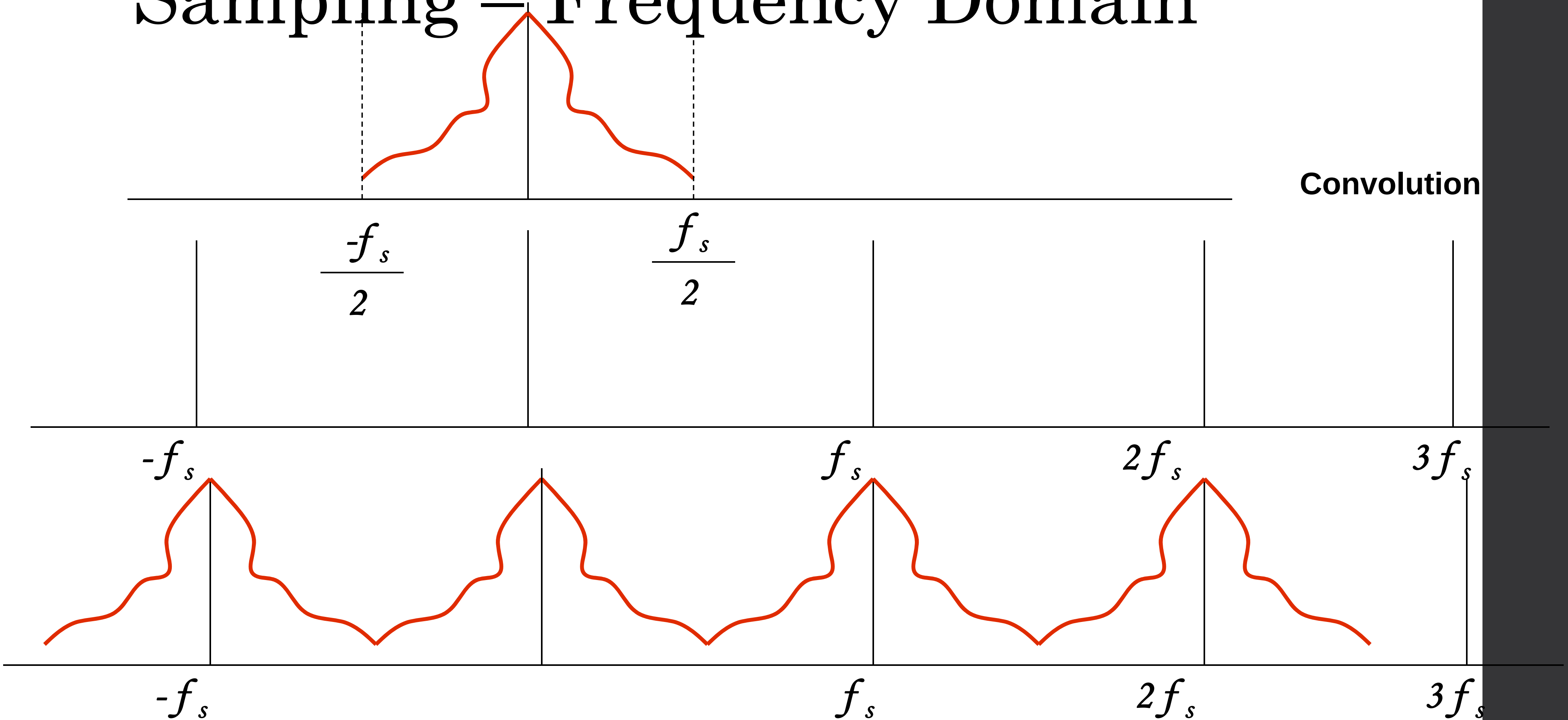


Aliasing

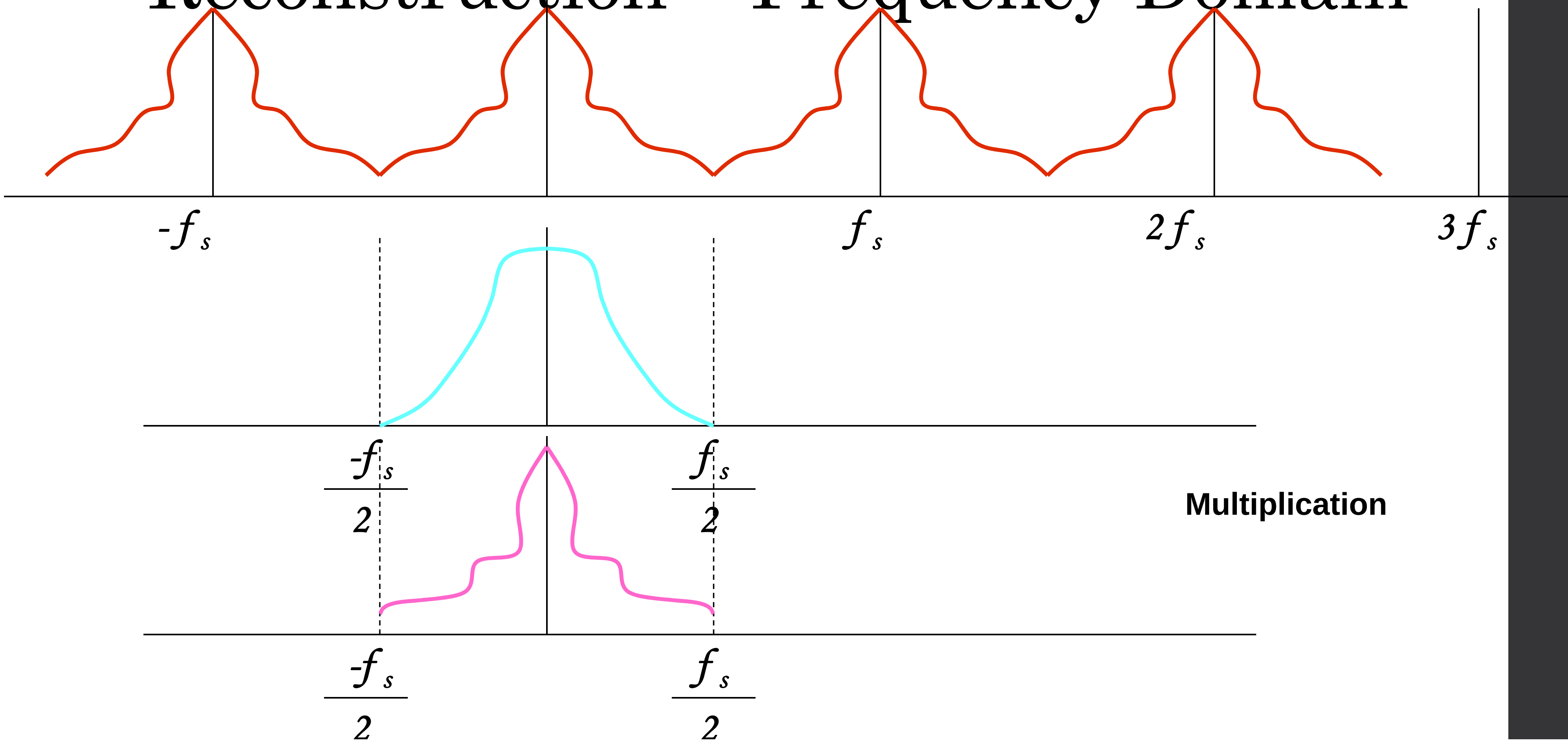




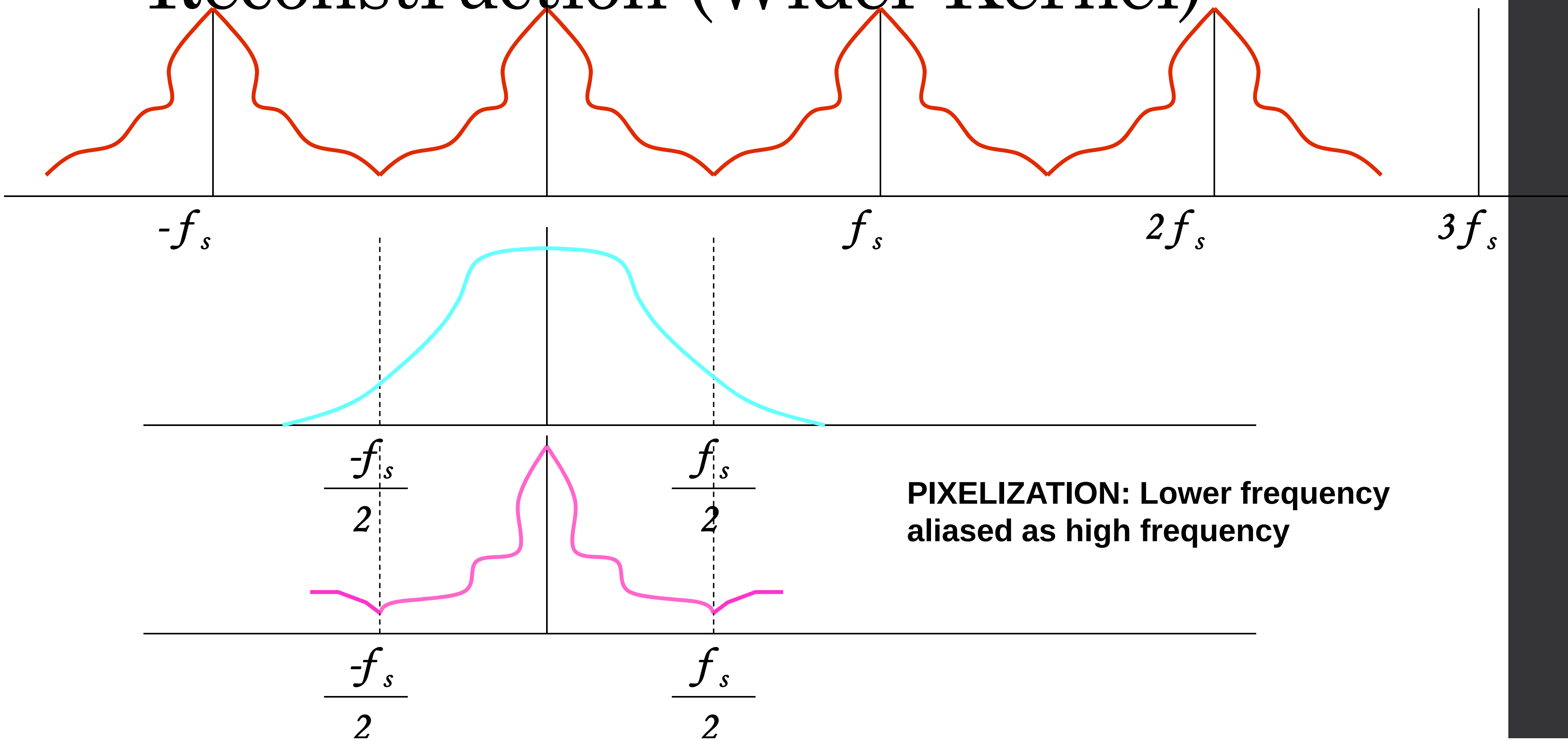
Sampling – Frequency Domain



Reconstruction – Frequency Domain

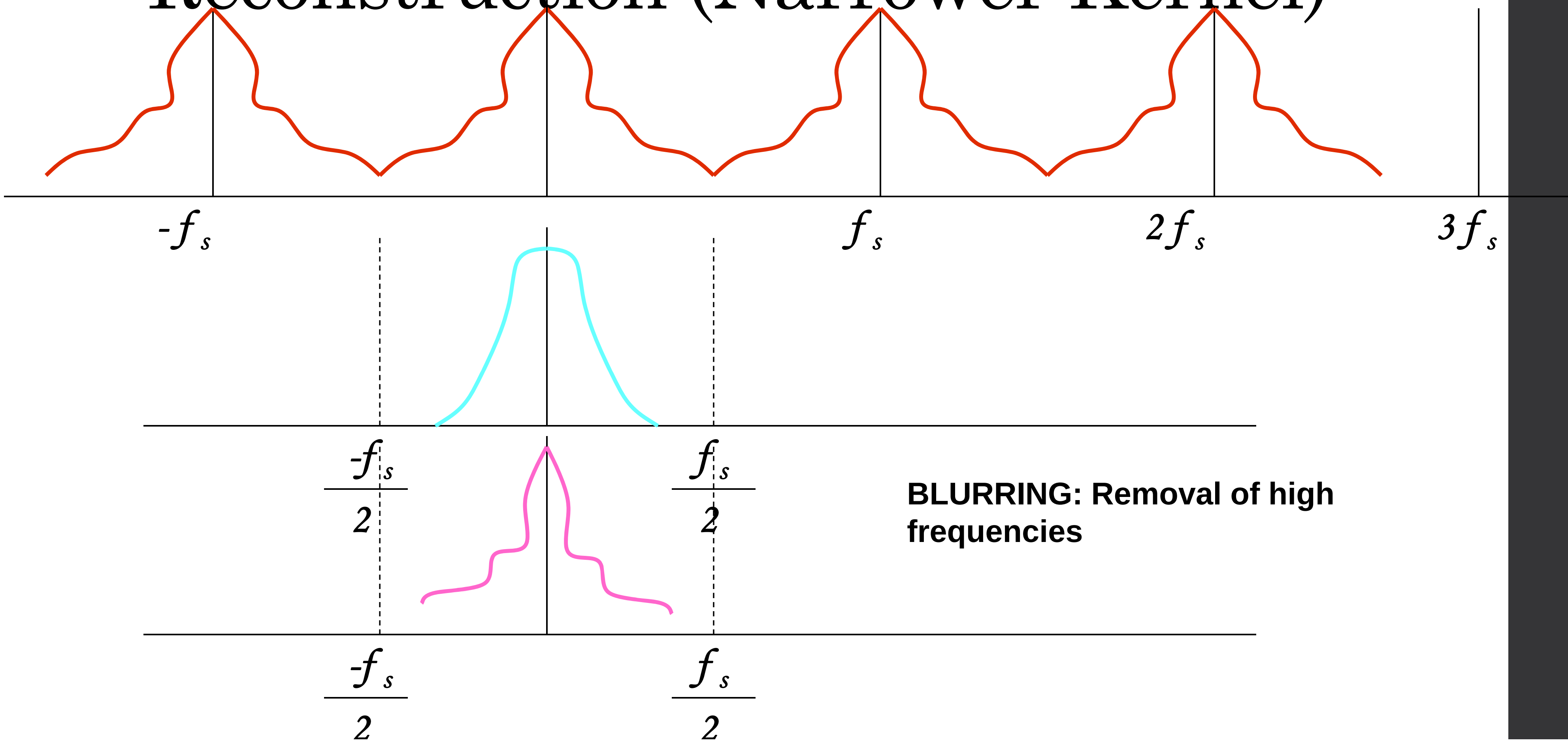


Reconstruction (Wider Kernel)



PIXELIZATION: Lower frequency aliased as high frequency

Reconstruction (Narrower Kernel)



BLURRING: Removal of high frequencies

Aliasing artifacts (Right Width)

Wider Spots (Lost high frequencies)

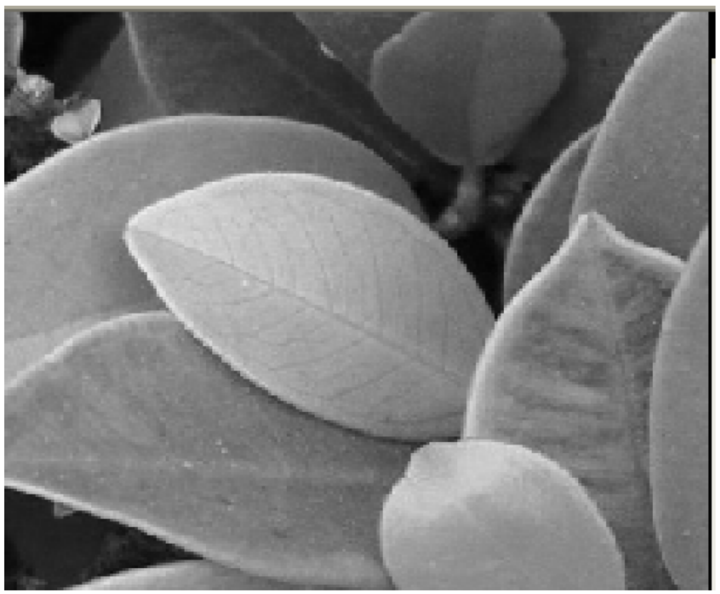
Narrow Width (Jaggies, insufficient sampling)

DFT extended to 2D : Axes

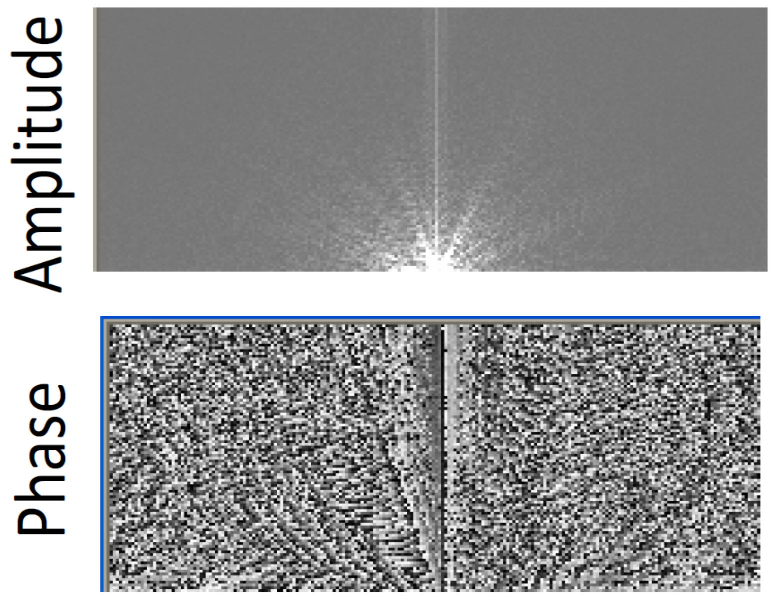
- Frequency
  - Only positive
- Orientation
  - 0 to 180
- Repeats in negative frequency
  - Just as in 1D

Example

Spatial Domain

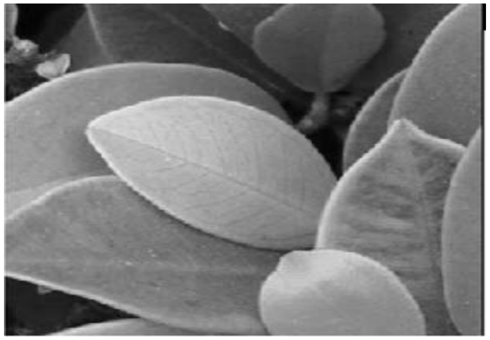


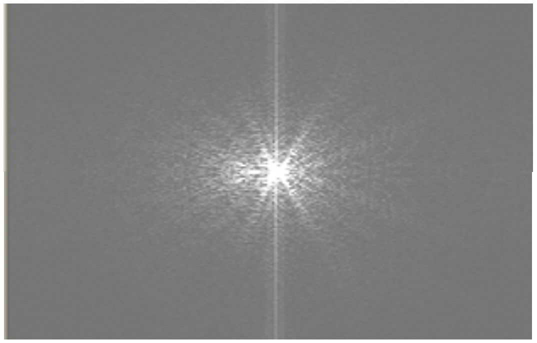
Frequency Domain



How it repeats?

- Just like in 1D
  - Even function for amplitude
  - Odd function for phase
- For amplitude
  - Flipped on the bottom





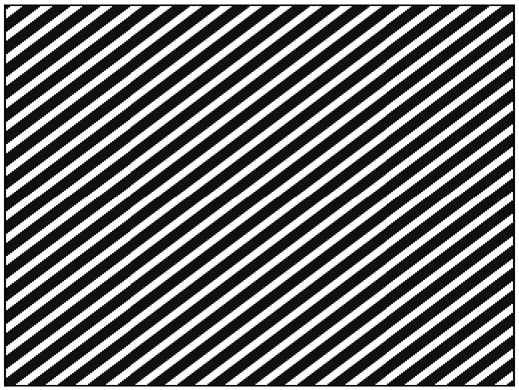
Why all the noise?

- Values much bigger than 255
- DC is often 1000 times more than the highest frequencies
- Difficult to show all in only 255 gray values

Mapping

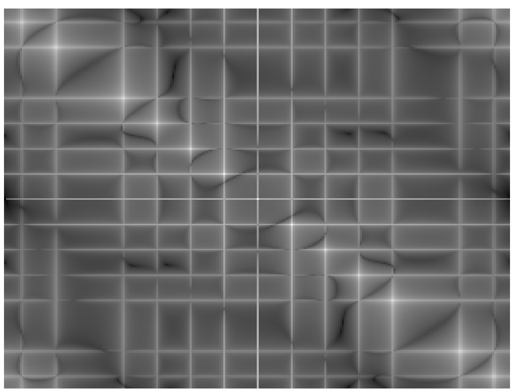
- Numerical value = ***i***
- Gray value = ***g***
- Linear Mapping is ***g = ki***
- Logarithmic mapping is ***g = k log (i)***
  - Compresses the range
  - Reduces noise
  - May still need thresholding to remove noise

Example



Original

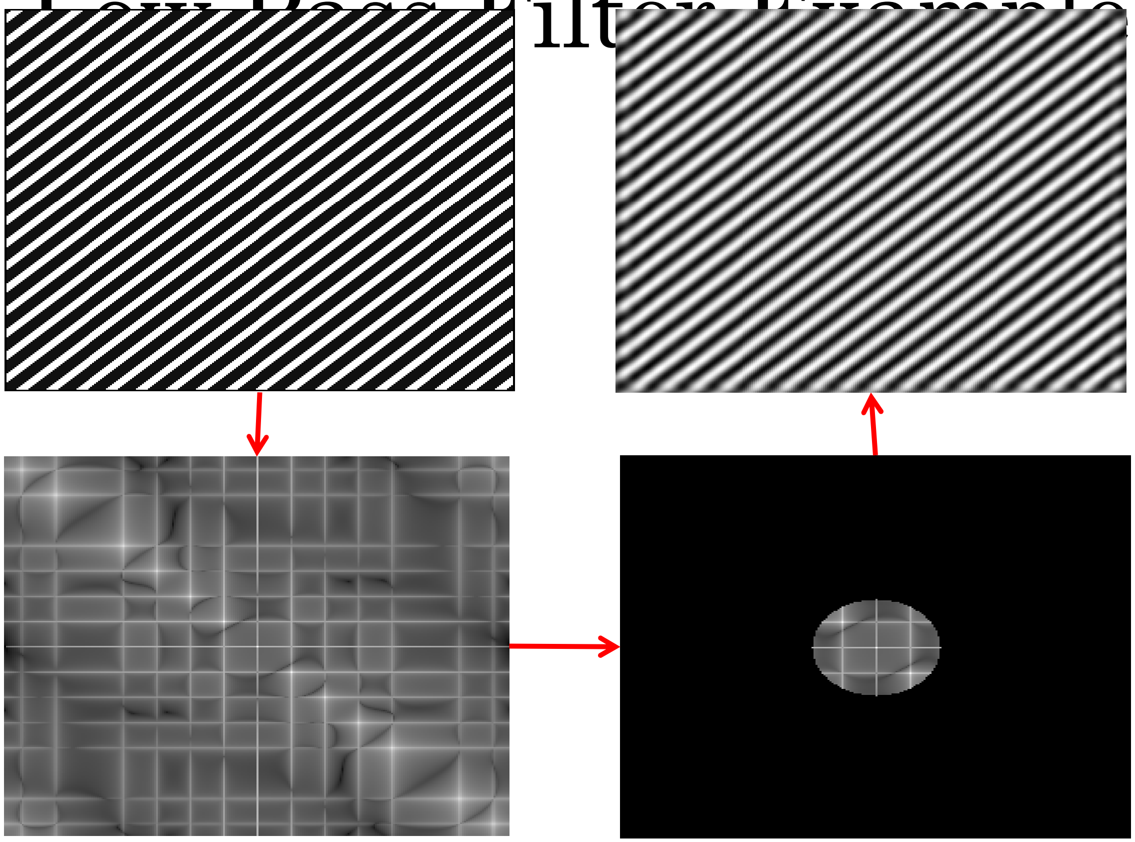
DFT Magnitude



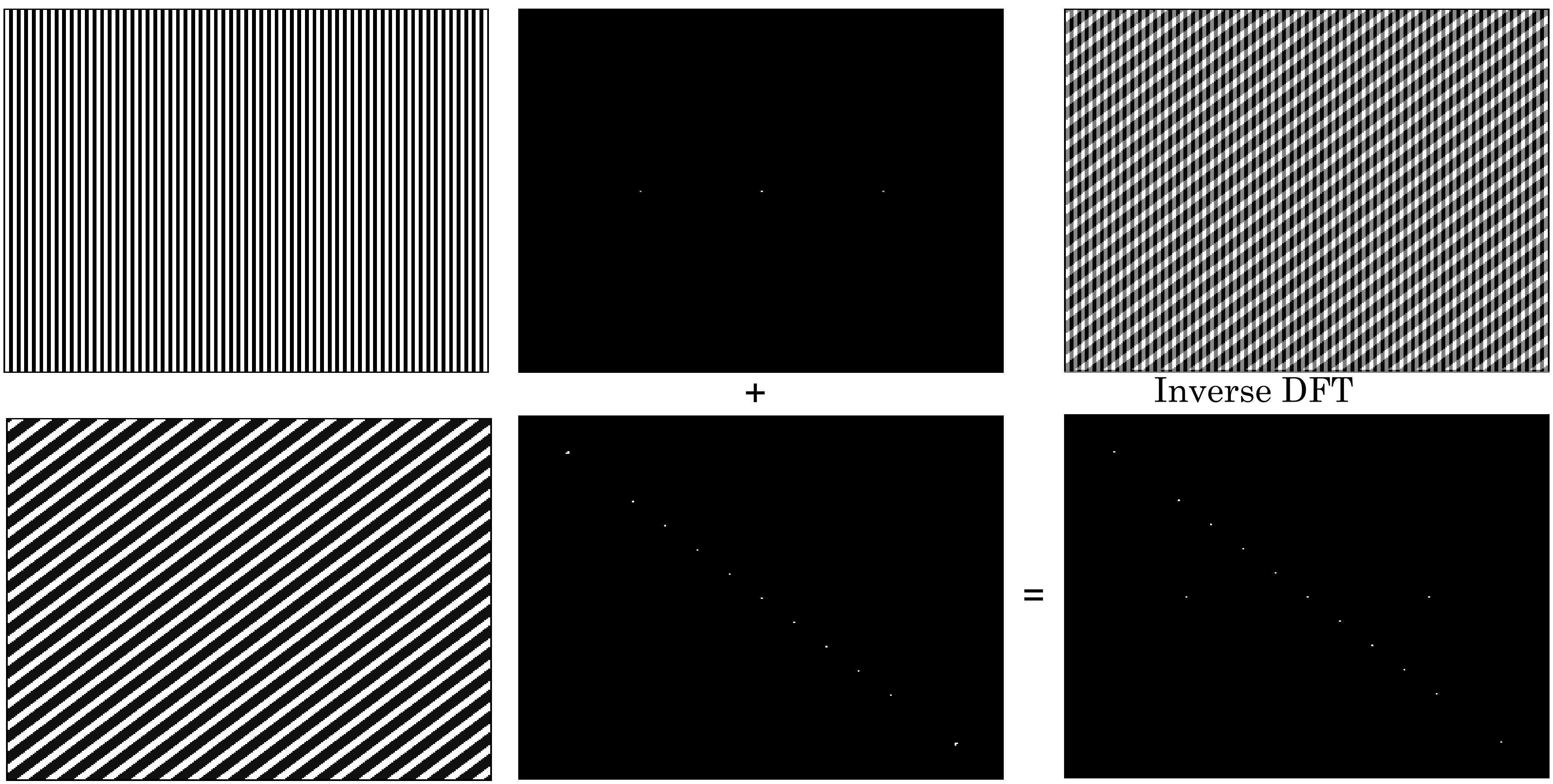
In Log scale

Post Thresholding

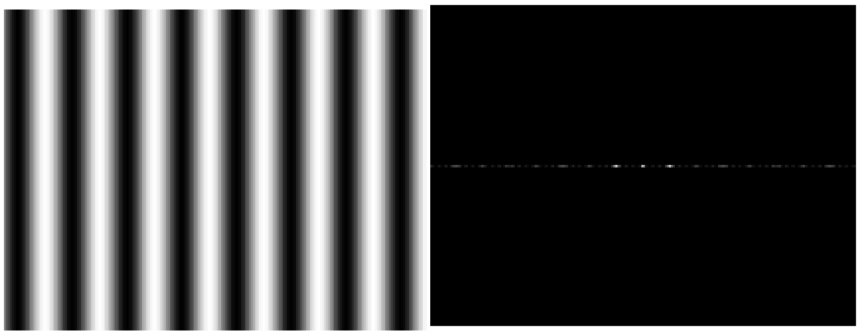
Low Pass Filter Example

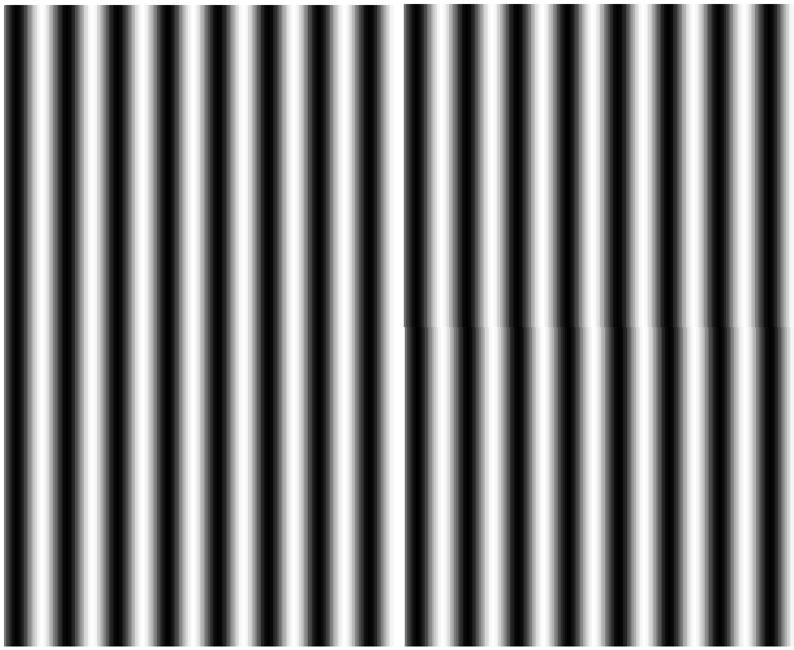


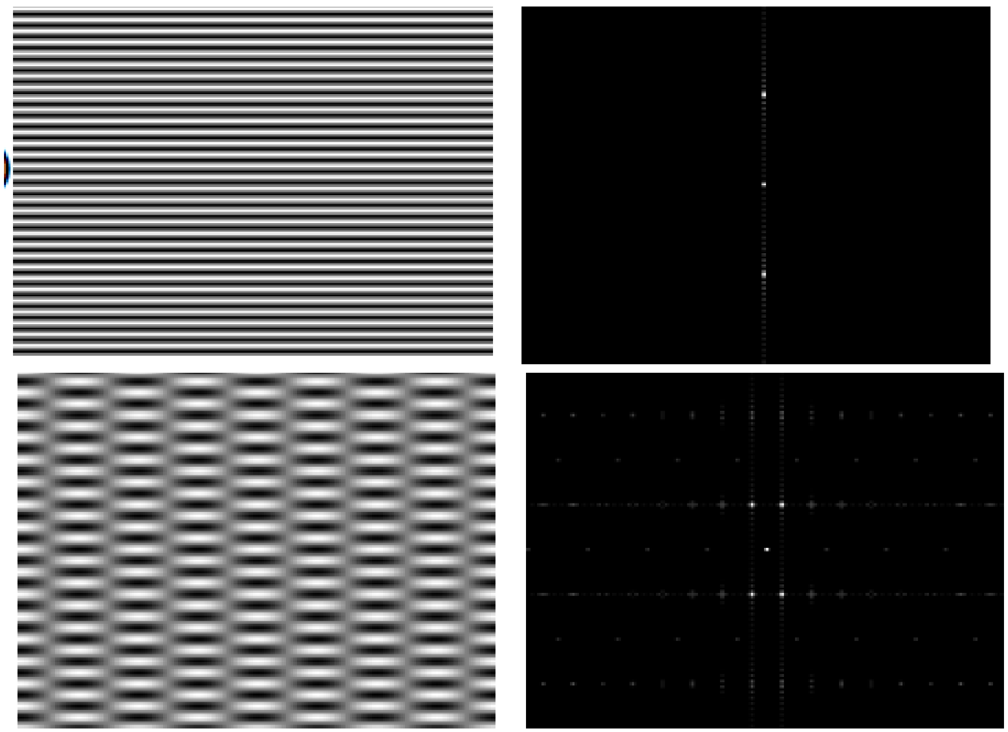
Additivity

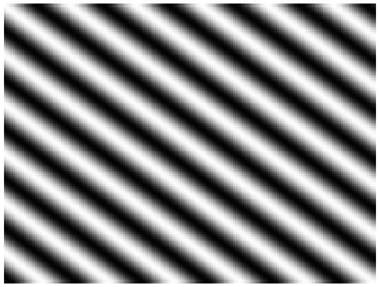
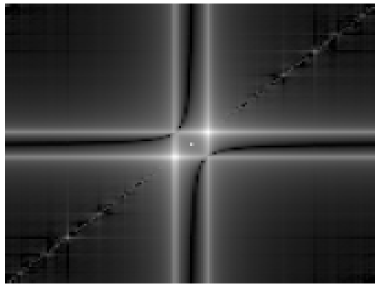
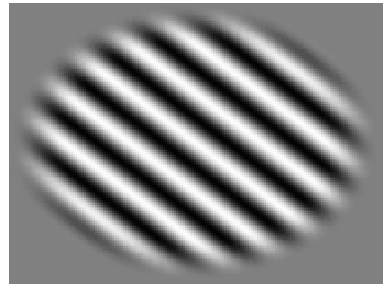
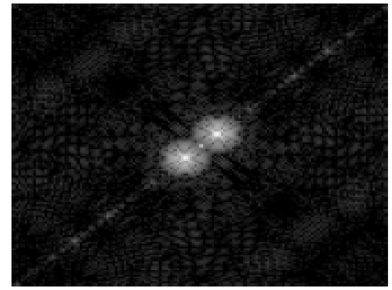
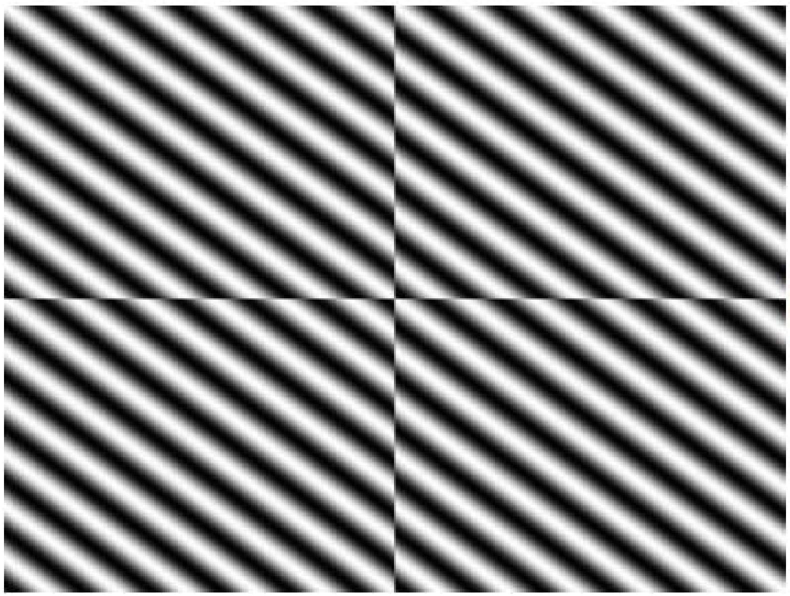
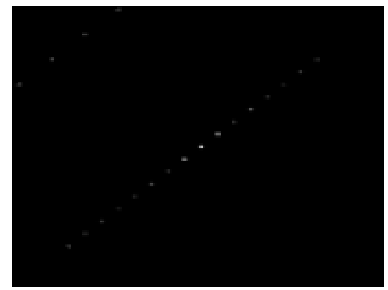


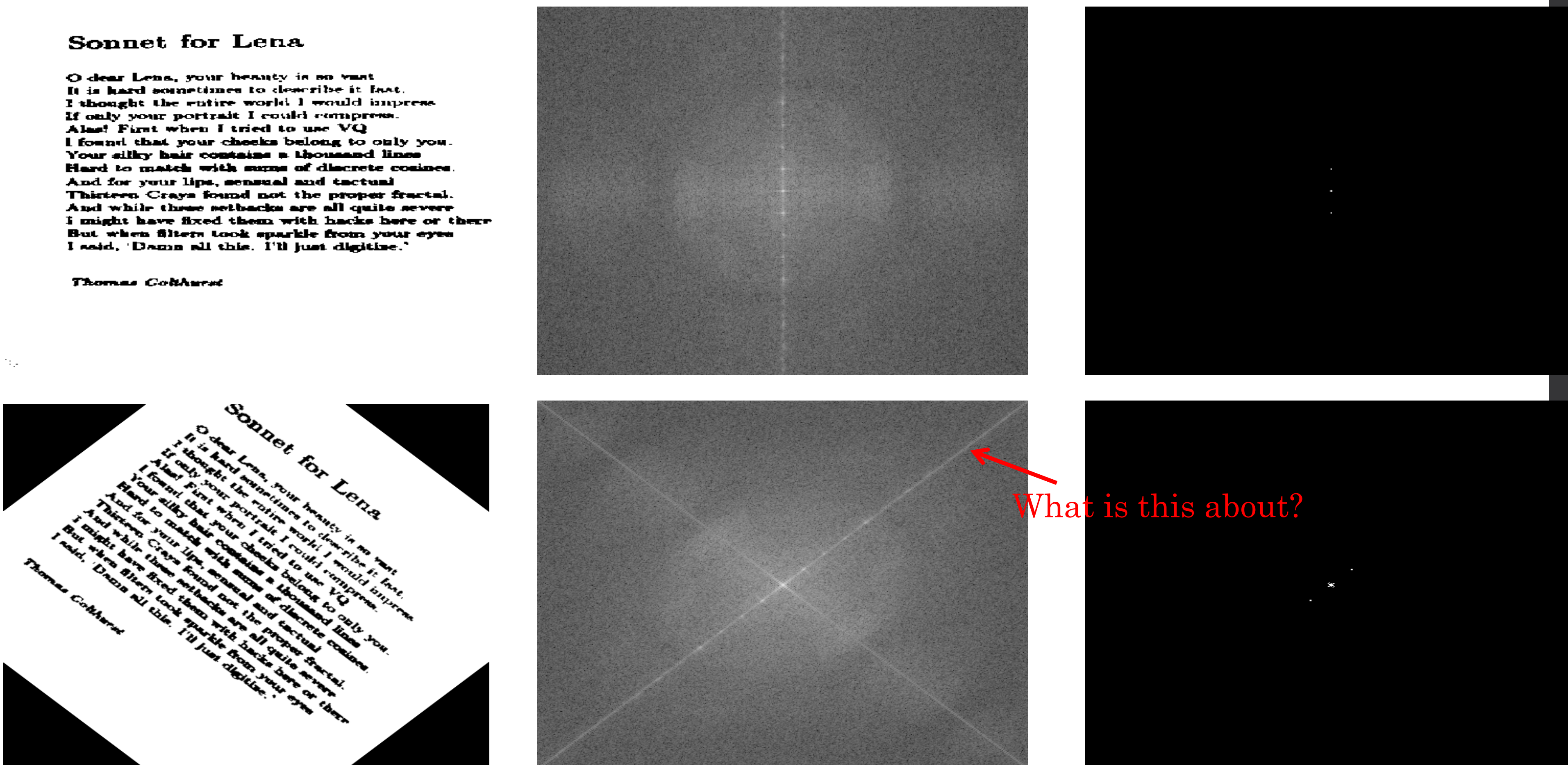
Nuances

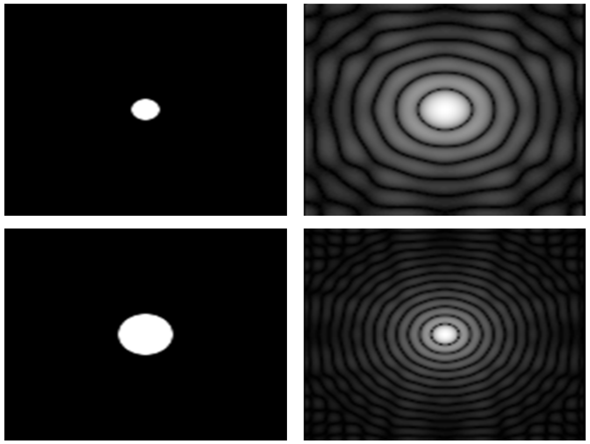


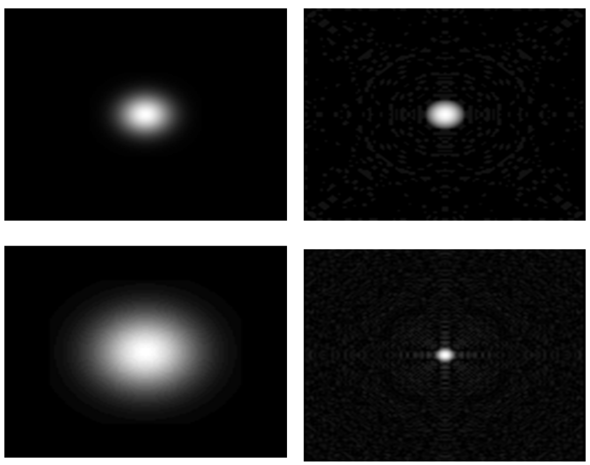


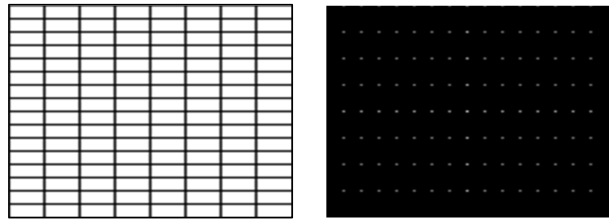


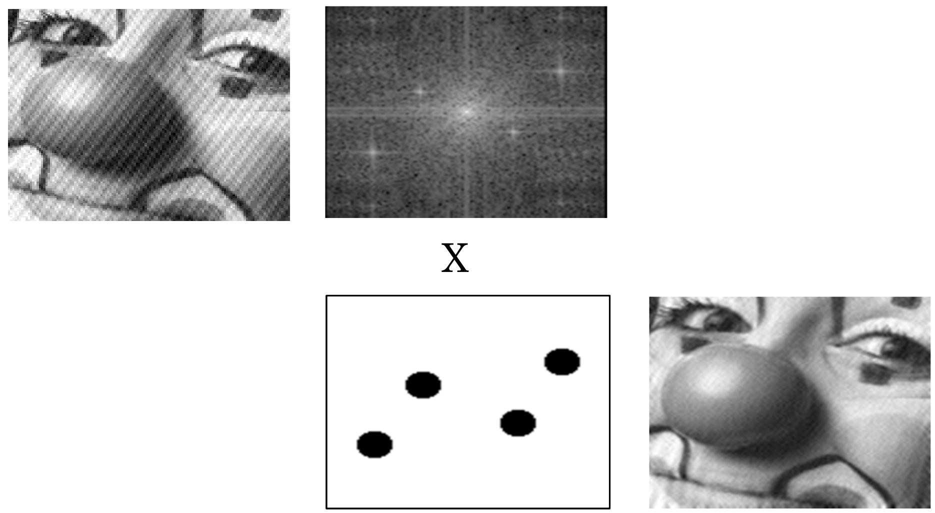







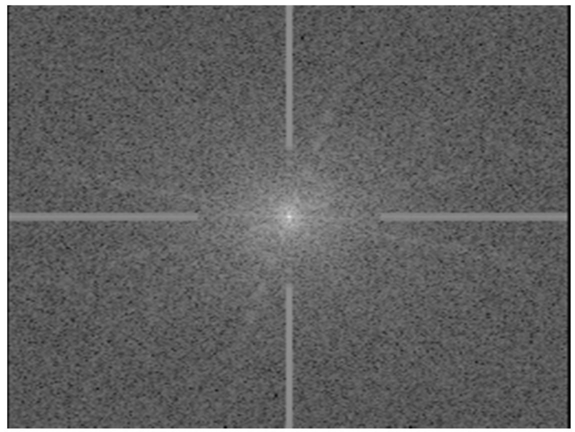

Sonnet for Lena
O dear Lena, your beauty is so vast
It is hard sometimes to describe it fast.
I thought the entire world I would impress
If only your portrait I could compress.
Alas! First when I tried to use VQ
I found that your cheeks belong to only you.
Your silky hair contains a thousand lines
Hard to match with sums of discrete cosines.
And for your lips, sensual and tactual
Thirteen Crays found not the proper fractal.
And while these setbacks are all quite severe
I might have fixed them with hacks here or there
But when filters took sparkle from your eyes
I said, 'Damn all this. I'll just digitize.'
Thomas Colthurst
What is this about?



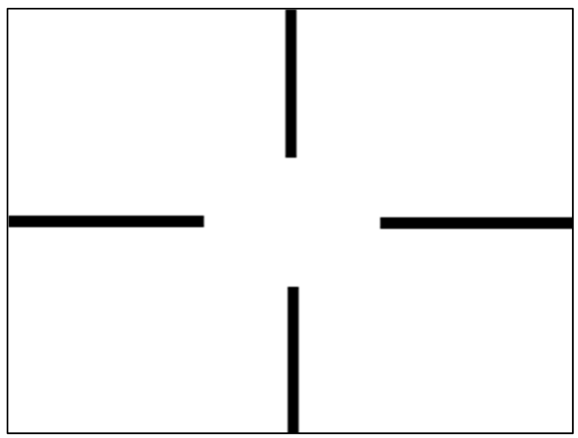




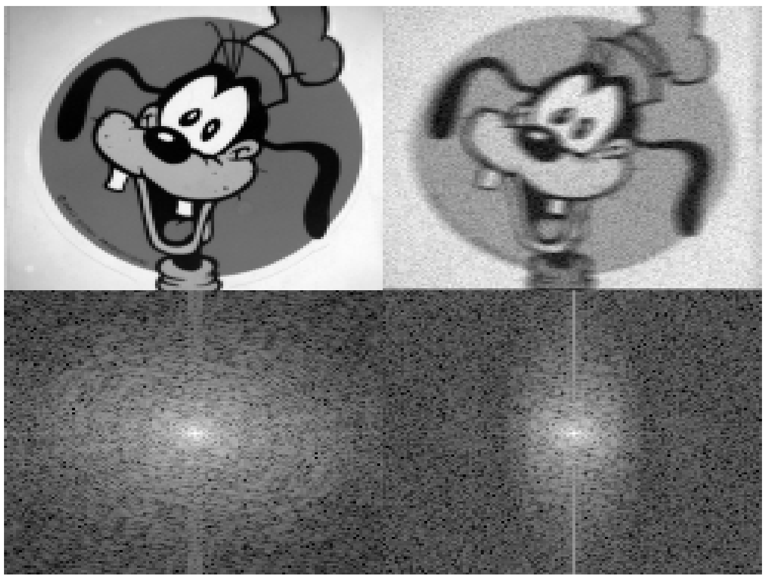

X

X

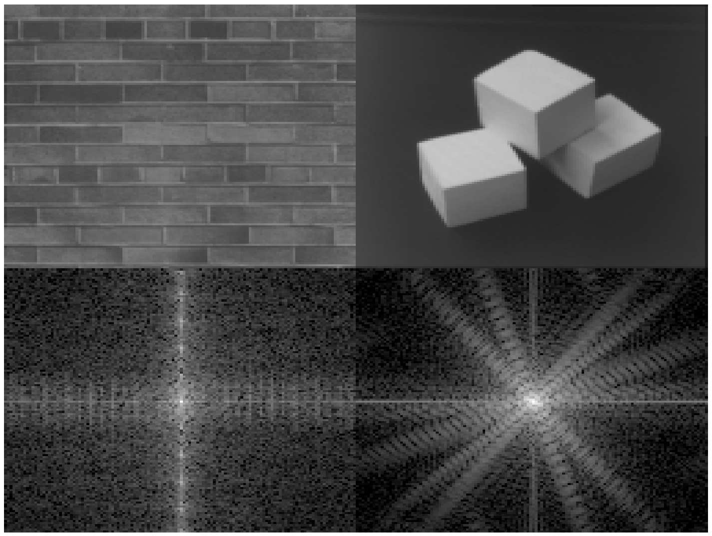



More examples: Blurring



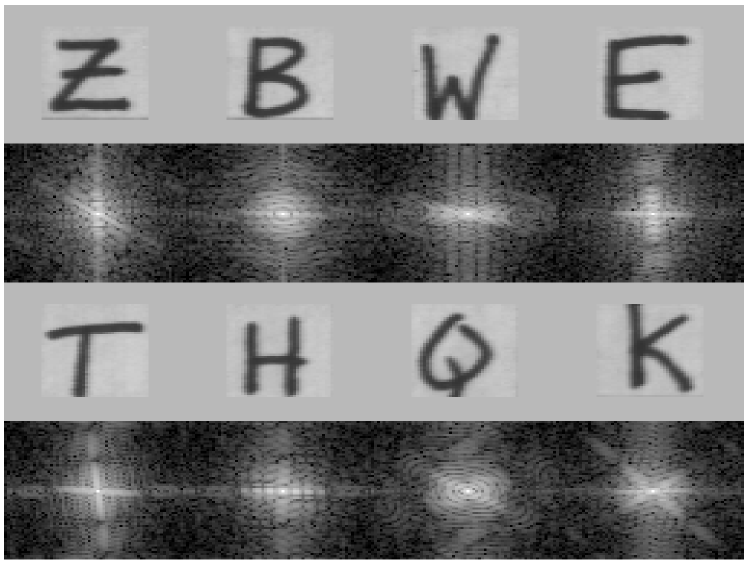
- Note energy reduced at higher frequencies
- What is direction of blur?
  - Horizontal
- Noise also added
  - DFT more noisy

More examples: Edges



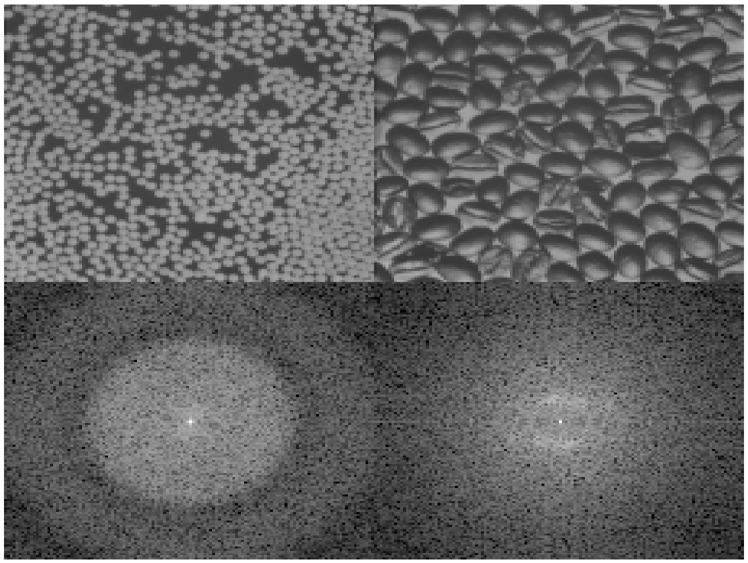
- Two direction edges on left image
  - Energy concentrated in two directions in DFT
- Multi-direction edges
  - Note how energy concentration synchronizes with edge direction

More examples: Letters



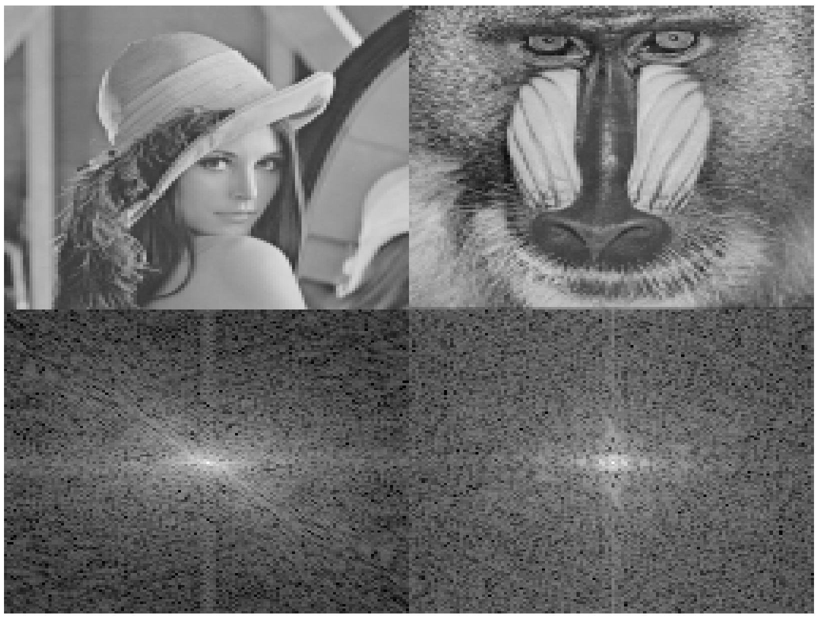
- DFTs quite different
  - Specially at low frequencies
- Bright lines perpendicular to edges
- Circular segments have circular shapes in DFT

More examples: Collections



- Concentric circle
  - Due to pallets symmetric shape
  - DFT of one pallet
    - Similar
- Coffee beans have no symmetry
  - Why the halo?
  - Illumination

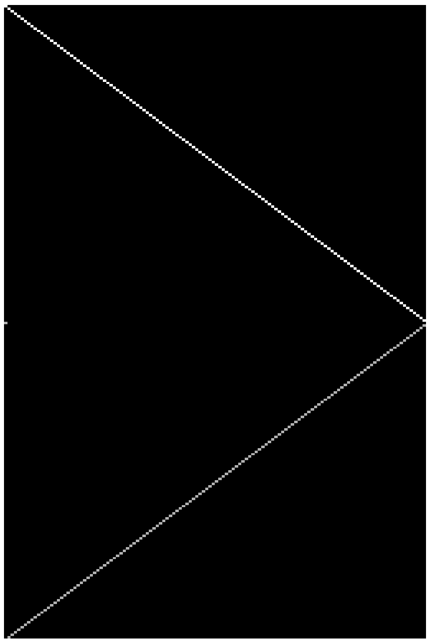
More examples: Natural Images



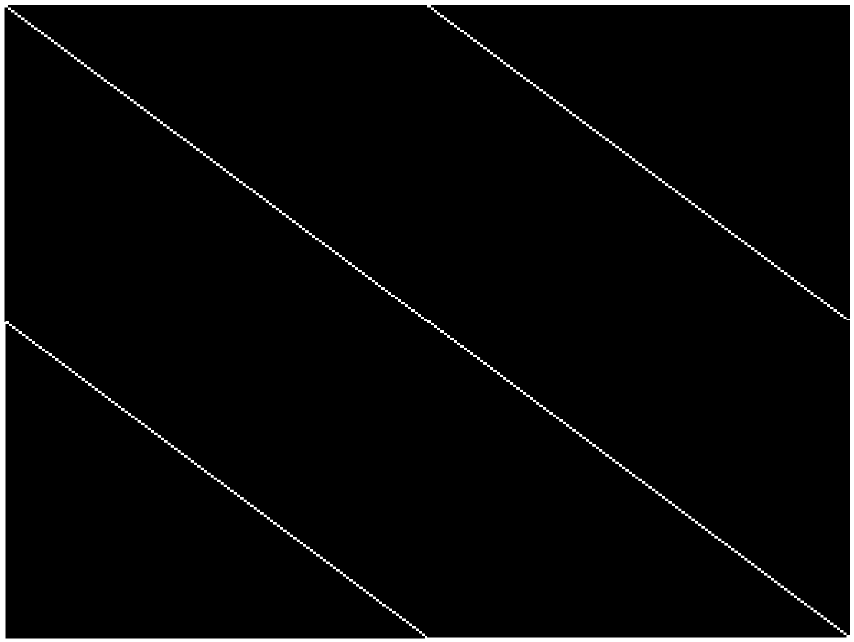
- Natural Images
- Why the diagonal line in Lena?
  - Strongest edge between hair and hat
- Why higher energy in higher frequencies in Mandril?
  - Hairs

More examples

Spatial



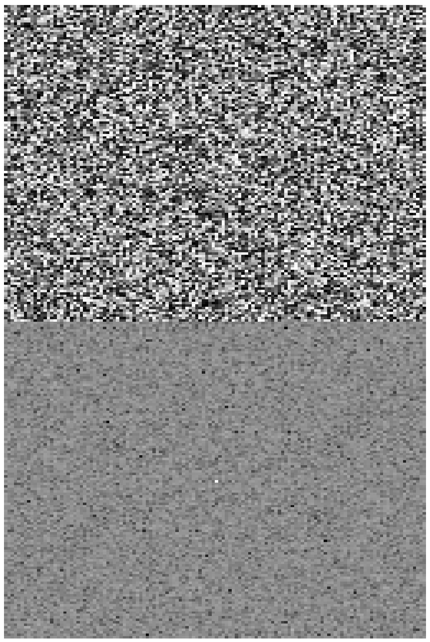
Frequency



- Repeatation makes perfect periodic signal
- Therefore perfect result perpendicular to it

More examples

Spatial



Frequency

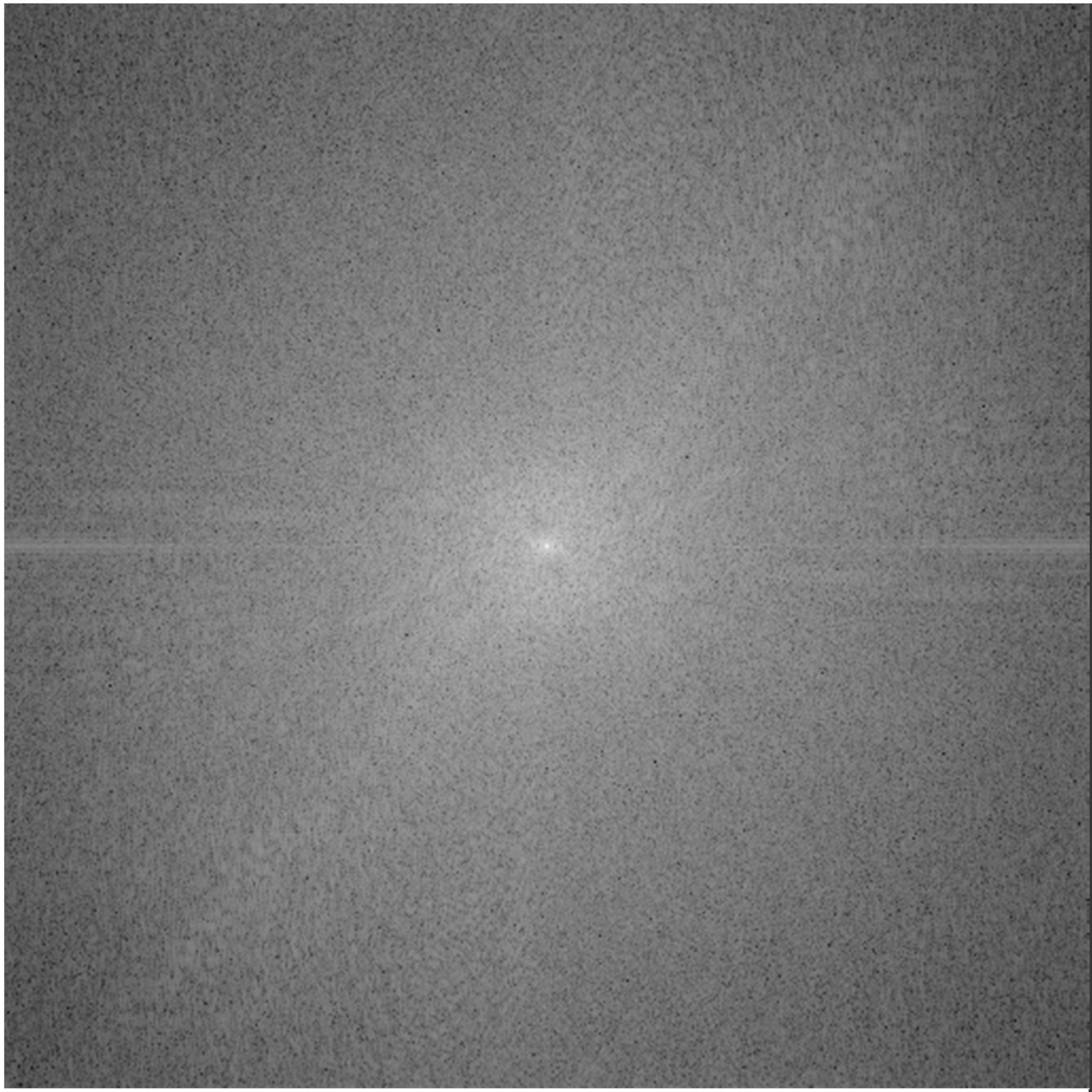
- Just a gray telling all frequencies
- Why the bright white spot in the center?

Amplitude

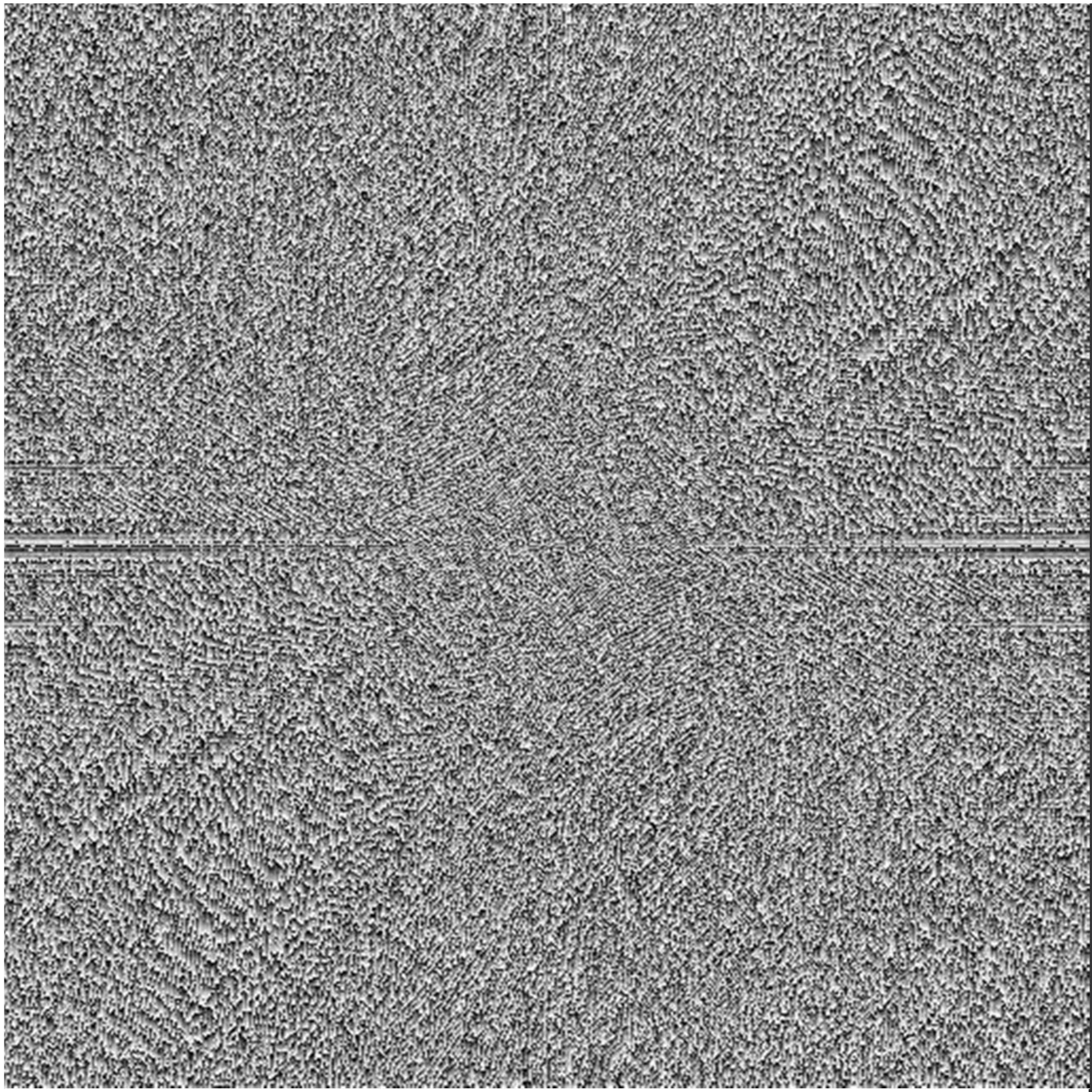
- **How** much details?
- Sharper details signify higher frequencies
- Will deal with this mostly

Cheetah

Magnitude

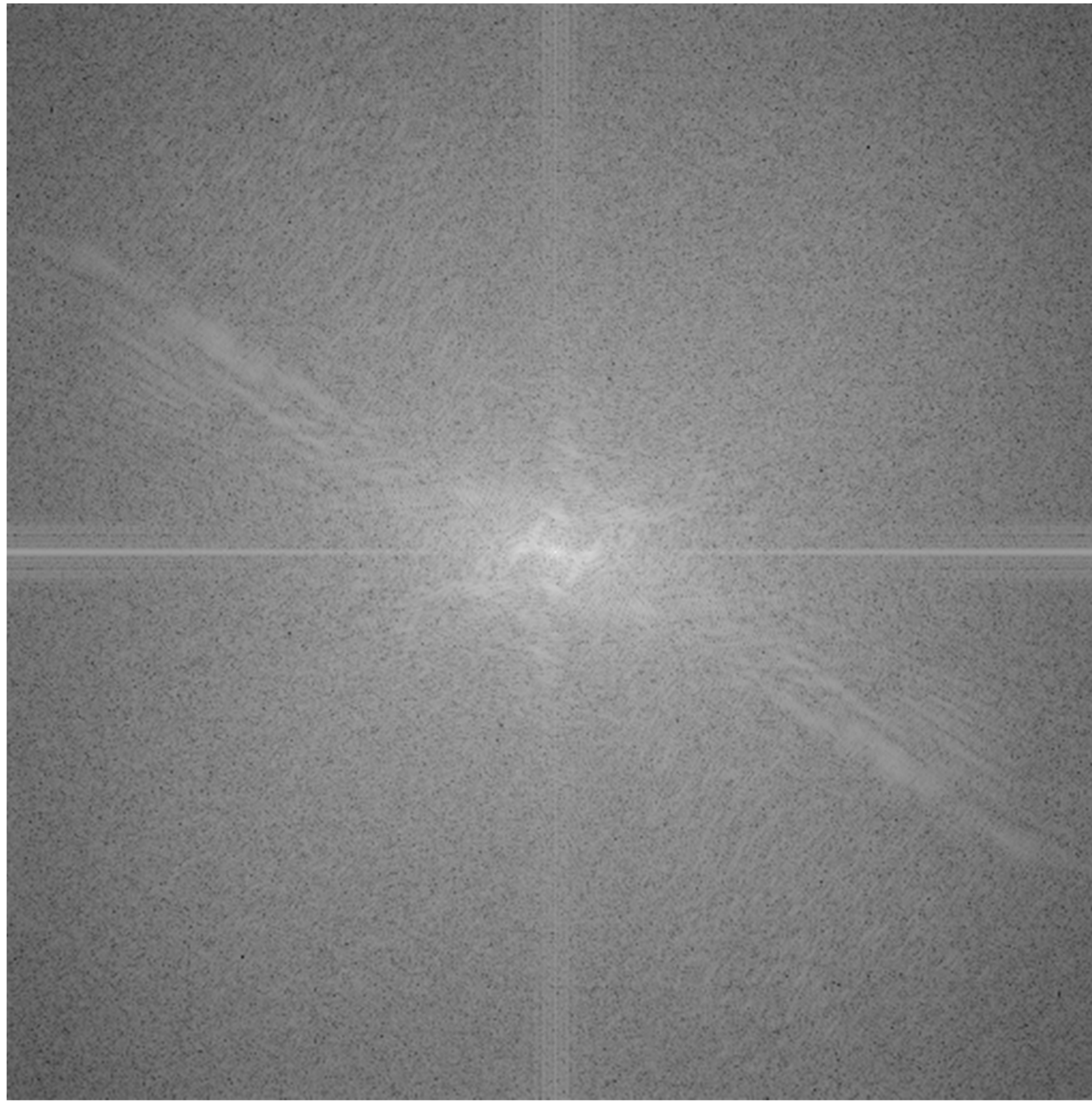


Phase

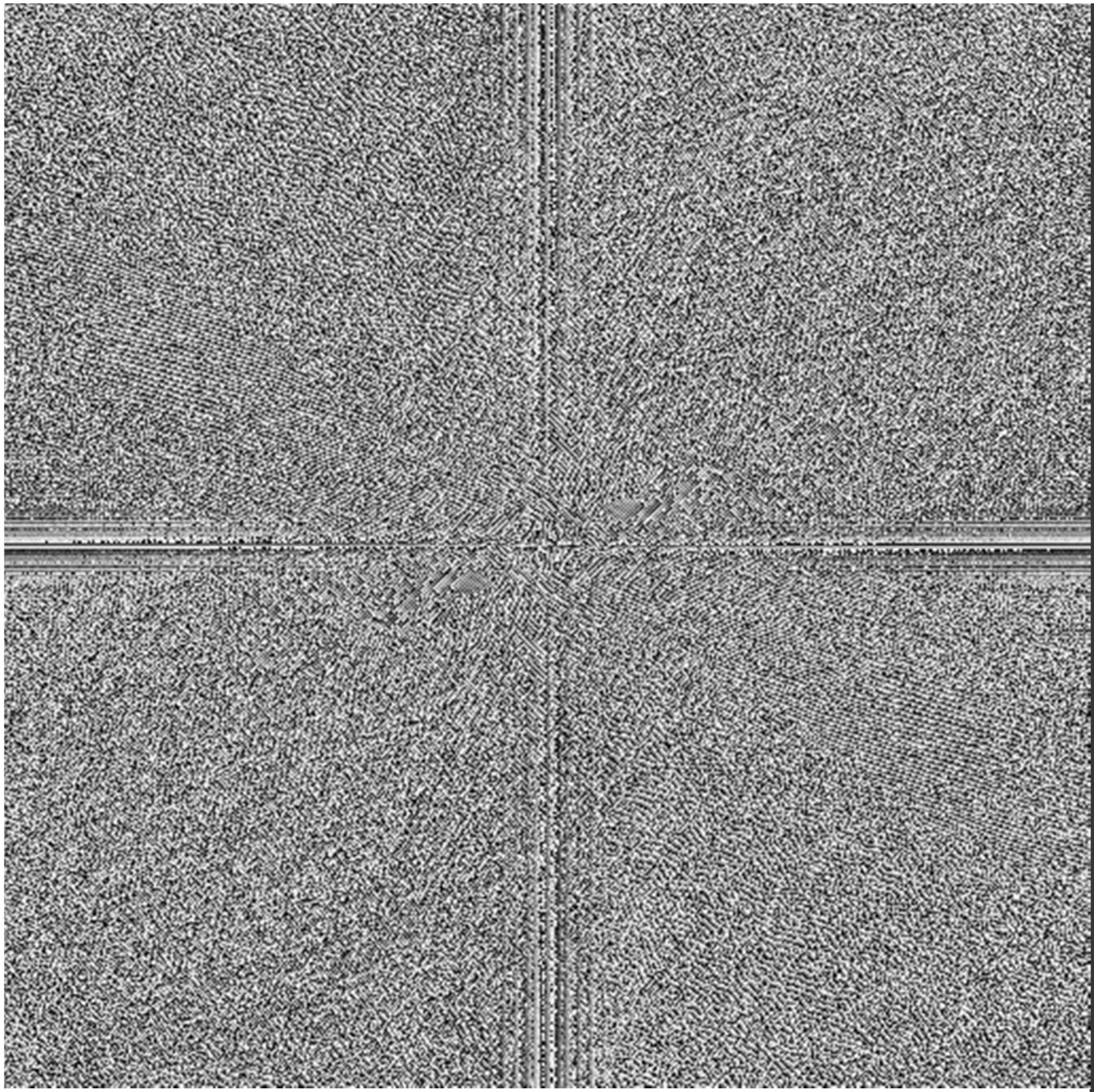


Zebra

Magnitude

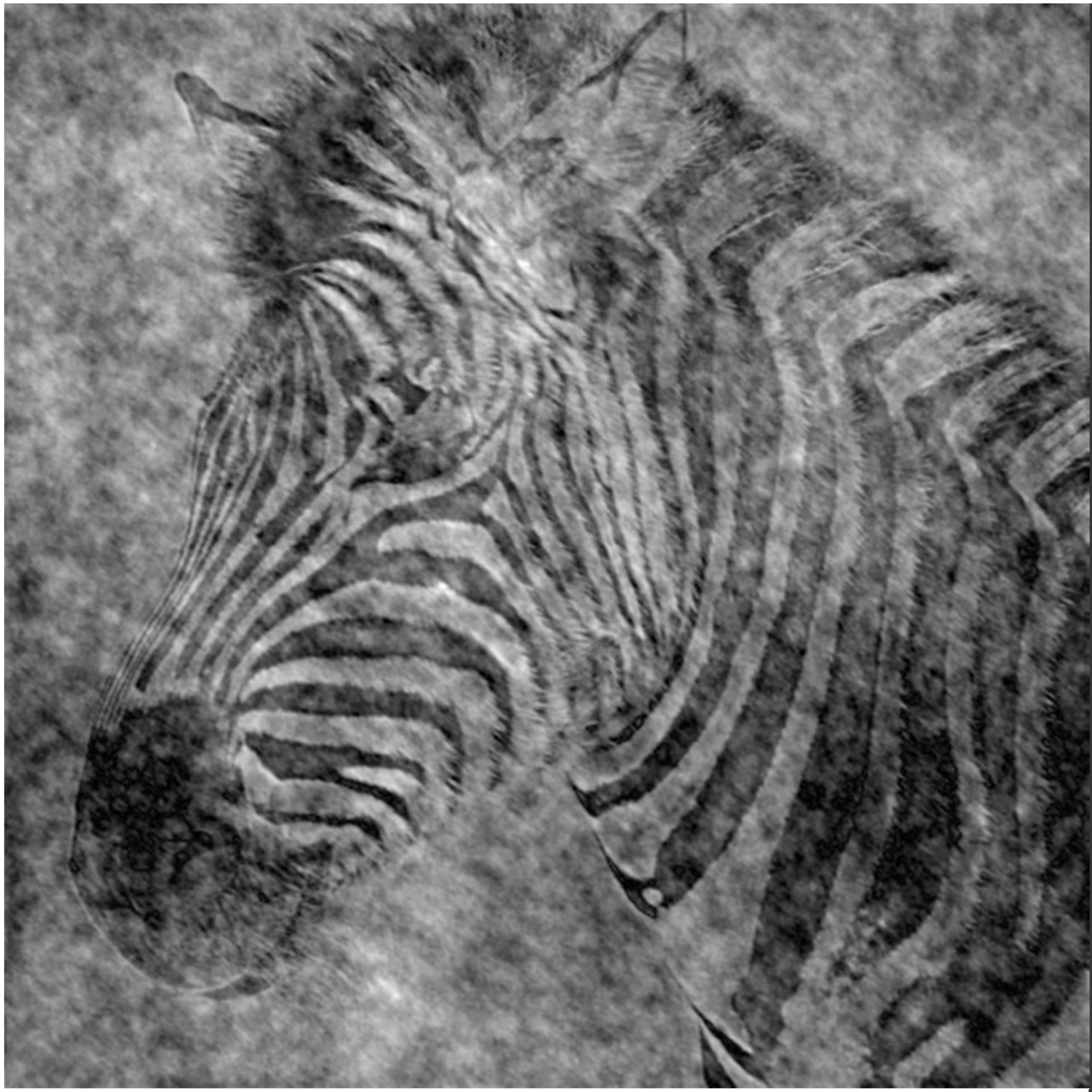


Phase



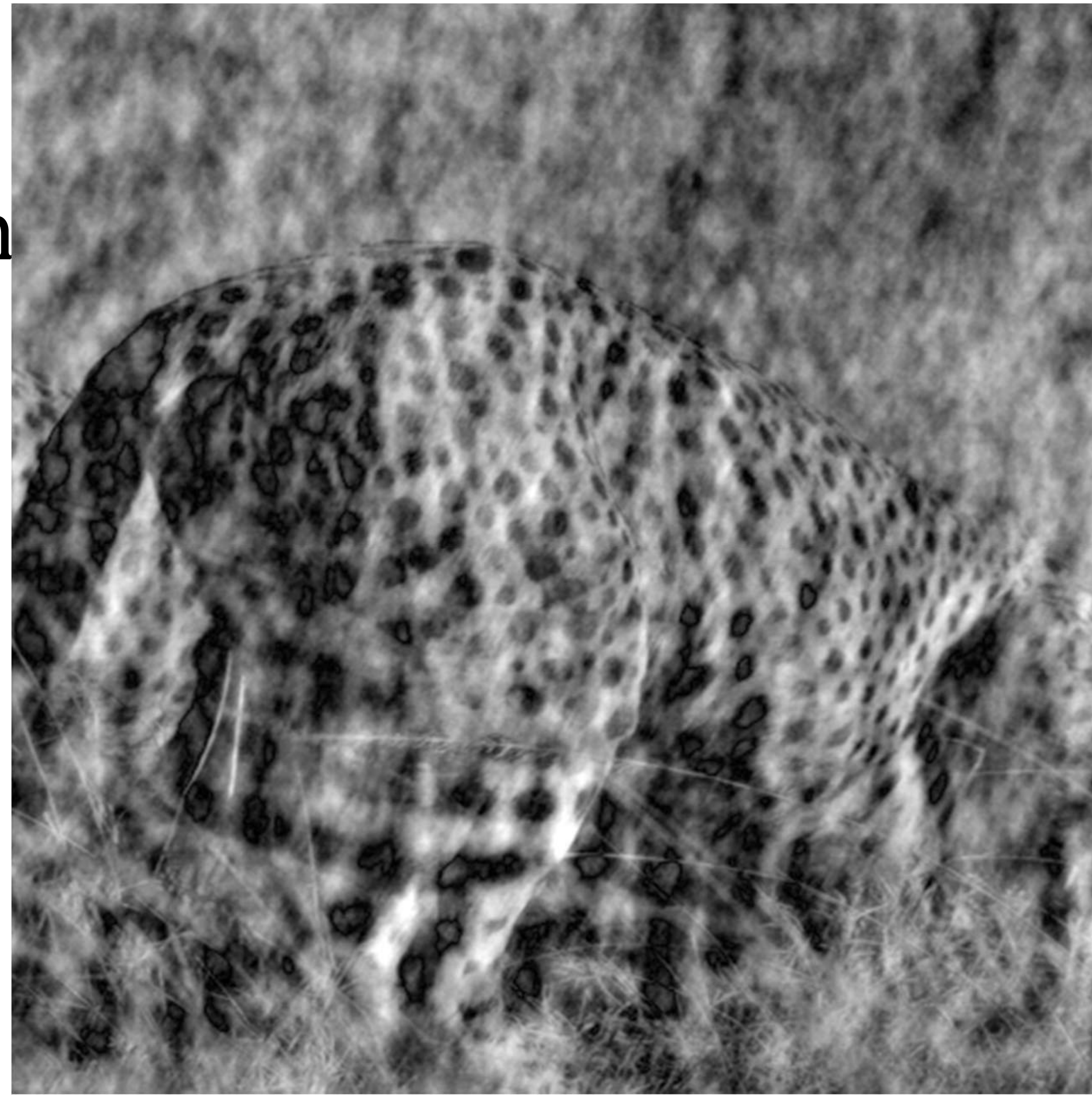
Reconstruction

- Cheetah Magnitude
- Zebra Phase



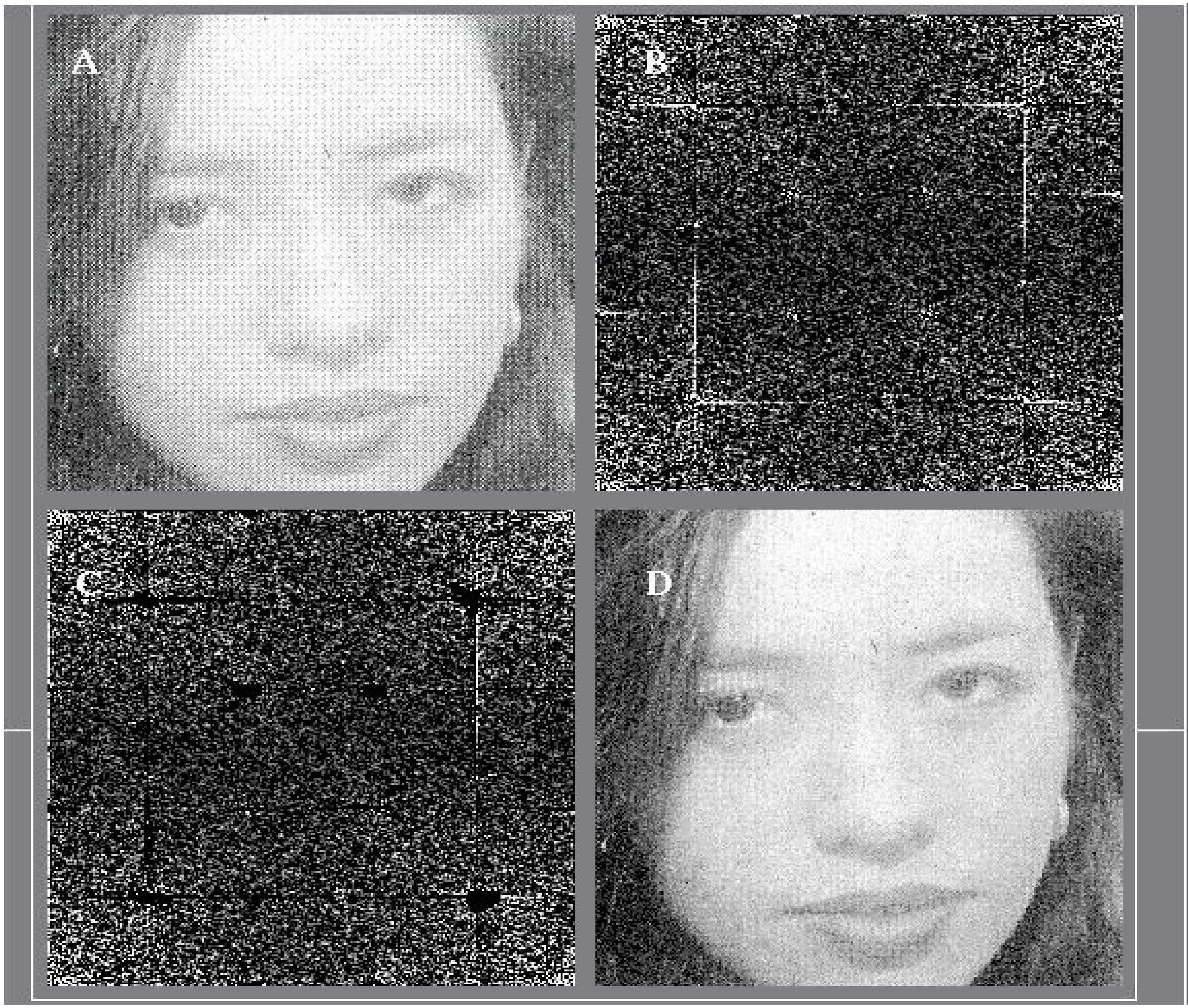
Reconstruction

- Zebra magnitude
- Cheetah phase

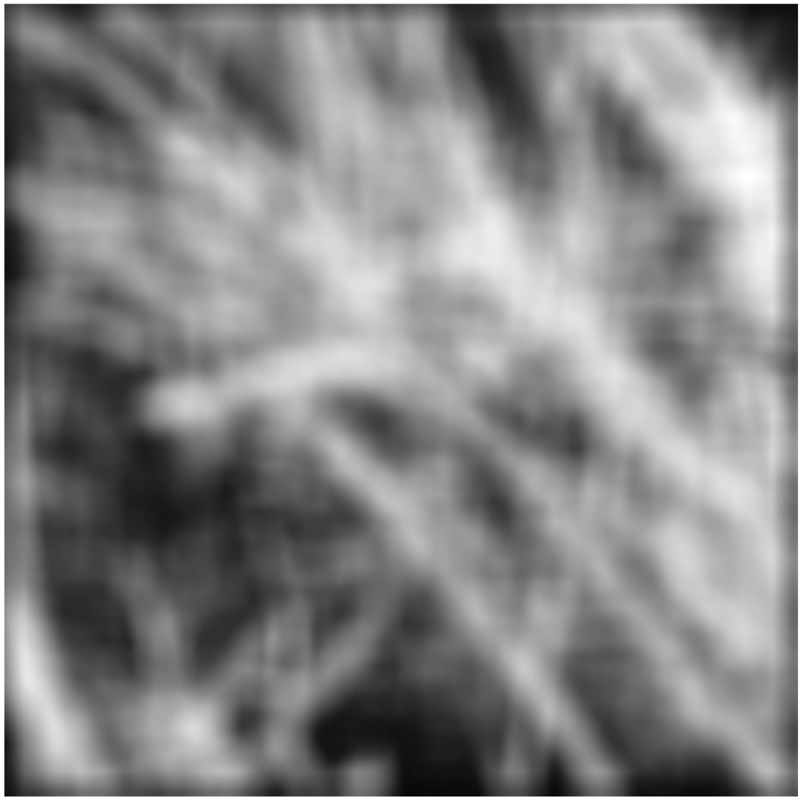


Uses – Notch Filter

Uses



Smoothing Box Filter



Smoothing Gaussian Filter