10/6 Visual Computing

▶ **Correlation**

Correlation is similar to doing the regular dot product in a sliding-window manner.

▶ **(Discrete) Fourier Transform**

Idea : Discrete signals can be represented as a linear combination of sine and cosine waves. The number of waves is at most the same as the number of samples.

DFT (Analysis)

x[0,1,2,...,N-1] → C[0,1,..,N/2] , S[0,1,...,N/2]

Signal in time(space) domain   Signal in frequency domain

←

IDFT (Synthesis)

x : N discrete signals

C : amplitude of N/2+1 cosine waves

S : amplitude of N/2+1 sine waves

(Actually it is N+2 waves, not N waves. But we are not going to talk about this discrepancy.)

C[k] : cosine wave with k cycles over the N samples

This gives "How many cycles per sample".

As k goes from 0 to N/2

k = 0, 1, 2, 3, ... N/2

frequency f is changed according to :

f = 0, k/N, (k+1)/N, ... N/2N

We define angular frequency as :

$\omega = 2\pi f$

Same goes with S[k]

Finally the signal can be represented as :

x[i] $= \sum_{k=0}^{N/2} C'[k] \cos 2\pi ik/N + \sum_{k=0}^{N/2} S'[k] \sin 2\pi ik/N$

where,

$C'[0] = Cx[0]/N \quad C'\left[\frac{N}{2}\right] = C\left[\frac{N}{2}\right]/N$

$C'[k] = \frac{C[k]}{N/2} \quad S'[k] = \frac{S[k]}{N/2}$

(This boundary condition is caused because the first and last point represents only the half of the interval. That is, the first represent interval of [0,1/2] and the last represent [N-2/1,N]).

Then, how do we get C[k] & S[k] at the first place?

→ Analysis (correlation with sine and cosine waves)

$C[k] = \sum_{i=0}^{N-1} x[i]\cos(2\pi ki)$

$S[k] = \sum_{i=0}^{N-1} x[i]\sin(2\pi ki)$

By $Acos(f) + Bsin(f) = \sqrt{A^2+B^2}cos(f+\theta)$, $\theta = tan^{-1}(\frac{B}{A})$

This means that x can be written as cum of (N/2+1) cosine waves.

Problems of phase..

1. When A = 0, phase is not defined.
2. When A = B = 1, phase = 45 ; when A = B = -1, phase = 45 as well.
   (it is supposed to be 135.)
   → ambiguity

Properties of DFT

1. x[i] → M[f] ⇒ kx[i] → kM[f] (M : magnitude)
   When we amplify the signal, Magnitude part of DFT is also amplified by same factor.

2. x_1[i] → C_1[f],S_1[f], x_2[i] → C_2[f], S_2[f]
   ⇒ x_1[i] + x_2[i] → C_1[f] + C_2[f], S_1[f] + S_2[f]
   When two or more signals are added up, the coefficient of sine,cosine are added up as well.

3. x[i] → M[f], θ[f]
   ⇒ x[i+s] → M[f], θ[f] + 2πfs

When the signal is shifted, phase part is linearly increases in terms of the shift.

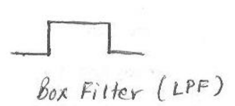
4. if the signal is symmetric, θ[f] = 0
   (because it can be decomposed into two out-of-phase signal)

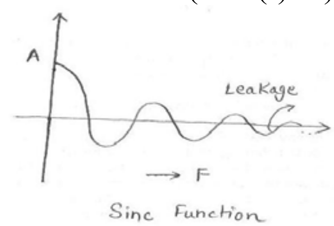
Duality between Time/Spatial Domain and Frequency

Time/Spatial Domain *Frequency Domain*

Impulse ----------- Constant

Box Filter ------------ Sinc Function (Sin (f) / f)





Sinc ----------- Box
Gaussian ------------ Gaussian

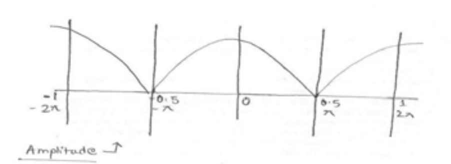
Comments:

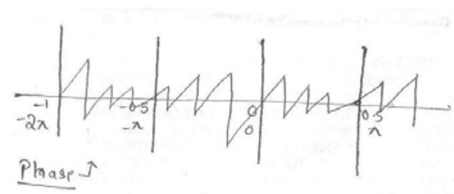
Theoretically , The sharper the cutoff the better is the filter so naturally Box Filter appears to be a nice LPF(Low Pass Filter), However it is not an ideal LPF (Low Pass Filter) as the Higher Frequency components are not completely cut off in the Frequency Response which is a Sinc Function. This is known as “Leakage”.

In the last class we studied the analysis equation where the magnitude M can be represented as a cosine function , A *COS(f),
M(f) = A*COS(f) = A* COS(-f)
Also A*COS (f) = A* COS (2*Π –f)) [here * means regular scalar multiplication]
Since M(f) = M(-f) ,therefore M(f) is an even function

Phase(□) -

Cos(f+ Π/4)=cos(-f-Π/4)
Hence Phase is an Odd Function





Aliasing

Let the signal X be in the range [0…N] and the convolution kernel H be in range [0…M] Then the convolution X*H will produce N+M-1 samples .These extra high frequency components shows up in the low frequency region and phenomenon is known as aliasing

Example: Given N=256, M=51

X*H will give M+N-1 = 306 samples in time domain but frequency domain will contain 256 samples