Set 5: Constraint Satisfaction Problems

ICS 271 Fall 2014

Kalev Kask

Outline

- **The constraint network model**
  - Variables, domains, constraints, constraint graph, solutions
- **Examples:**
  - graph-coloring, 8-queen, cryptarithmetic, crossword puzzles, vision problems,scheduling, design
- **The search space and naive backtracking,**
- **The constraint graph**
- **Consistency enforcing algorithms**
  - arc-consistency, AC-1,AC-3
- **Backtracking strategies**
  - Forward-checking, dynamic variable orderings
- **Special case: solving tree problems**
- **Local search for CSPs**

Constraint satisfaction problems (CSPs)

Standard search problem:

state is a "black box"—any old data structure
that supports goal test, eval, successor

CSP:

state is defined by *variables* X_i with *values* from *domain* D_i

goal test is a set of *constraints* specifying
allowable combinations of values for subsets of variables

Simple example of a *formal representation language*

Allows useful *general-purpose* algorithms with more power
than standard search algorithms

Constraint Satisfaction

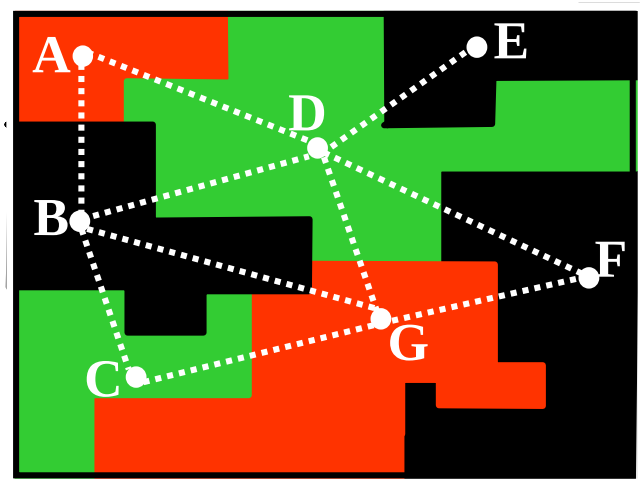
Example: map coloring

Variables - countries (A,B,C,etc.)

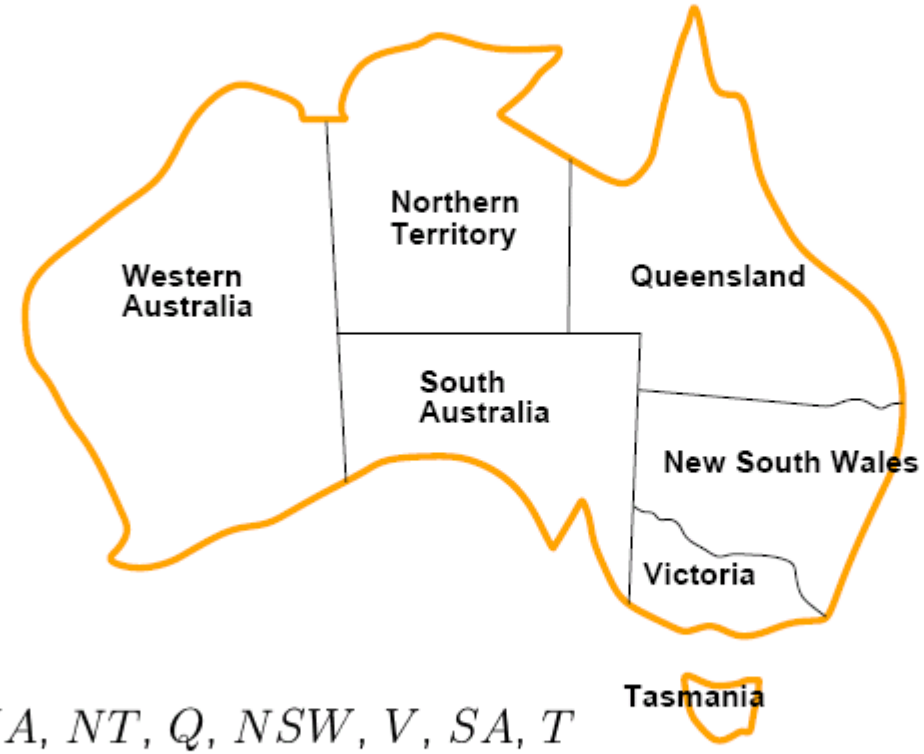
Values - colors (e.g., red, green, yellow)

Constraints: $A \neq B$, $A \neq D$, $D \neq E$, *etc.*

A	B
red	green
red	yellow
green	red
green	yellow
yellow	green
yellow	red



Example: Map-Coloring



Variables WA, NT, Q, NSW, V, SA, T

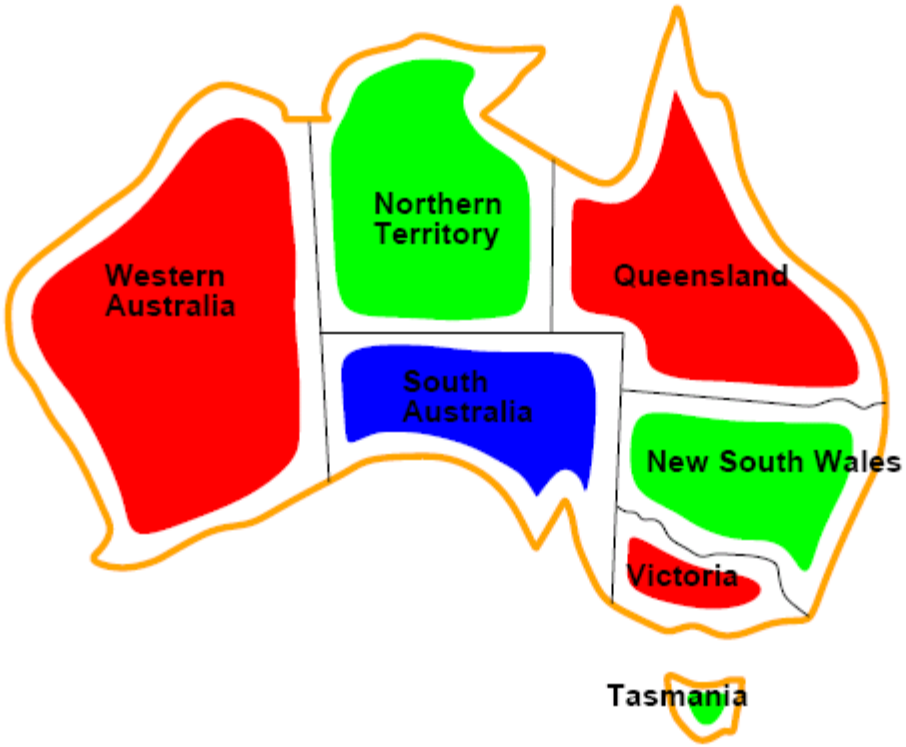
Domains $D_i = \{red, green, blue\}$

Constraints: adjacent regions must have different colors

e.g., $WA \neq NT$ (if the language allows this), or

$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$

Example: Map-Coloring contd.

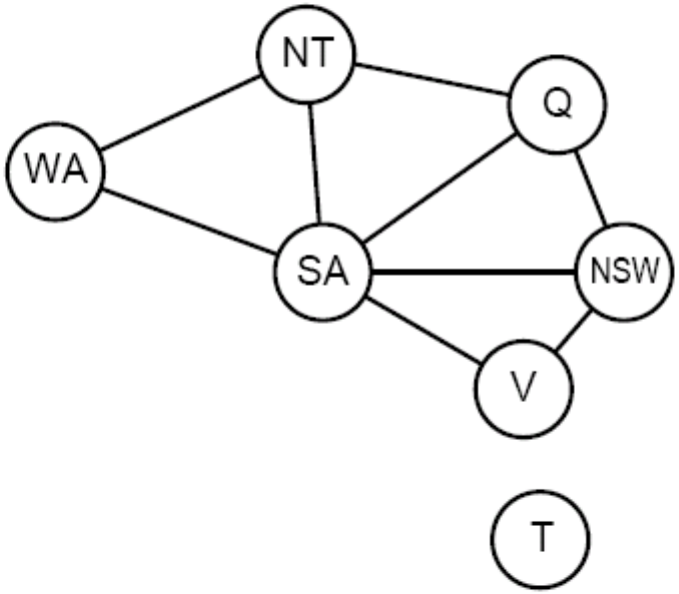


Solutions are assignments satisfying all constraints, e.g.,
$\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}$

Constraint graph

Binary CSP: each constraint relates at most two variables

Constraint graph: nodes are variables, arcs show constraints



General-purpose CSP algorithms use the graph structure
to speed up search. E.g., Tasmania is an independent subproblem!

Sudoku

		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	
		9			4	5	8	1
			3		2	9		

Each row, column and major block must be all different

"Well posed" if it has unique solution: 27 constraints

Sudoku

Constraint propagation

		2	4		6			
8	6	5	1			2		
	1				8	6		9
9				4		8	6	
	4	7				1	9	
	5	8		6				3
4		6	9				7	2
		9			4	5	8	1
			3		2	9		

- **Variables: 81 slots**
- **Domains = {1,2,3,4,5,6,7,8,9}**
- **Constraints:**
  - **27 not-equal**

Each row, column and major block must be all different

"Well posed" if it has unique solution: 27 constraints

Varieties of CSPs

Discrete variables

finite domains; size d $\Rightarrow$ $O(d^n)$ complete assignments

- ♢ e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)

infinite domains (integers, strings, etc.)

- ♢ e.g., job scheduling, variables are start/end days for each job
- ♢ need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- ♢ linear constraints solvable, nonlinear undecidable

Continuous variables

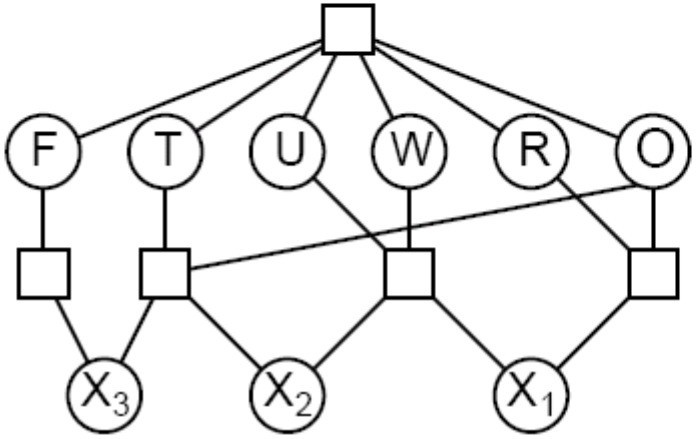
- ♢ e.g., start/end times for Hubble Telescope observations
- ♢ linear constraints solvable in poly time by LP methods

Varieties of constraints

- **Unary constraints involve a single variable,**
  - e.g., SA ≠ green
  -
- **Binary constraints involve pairs of variables,**
  - e.g., SA ≠ WA
  -
- **Higher-order constraints involve 3 or more variables,**
  - e.g., cryptarithmetic column constraints
  -

Example: Cryptarithmetic

$$\begin{array}{r} \ T\ W\ O \\ +\ T\ W\ O \\ \hline F\ O\ U\ R \end{array}$$



Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints

alldiff(F, T, U, W, R, O)

$O + O = R + 10 \cdot X_1$, etc.

Real-world CSPs

Assignment problems
    e.g., who teaches what class

Timetabling problems
    e.g., which class is offered when and where?

Hardware configuration

Spreadsheets

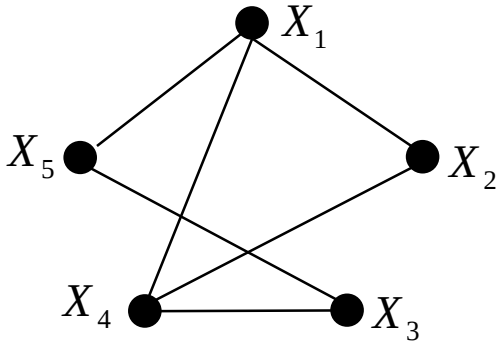
Transportation scheduling

Factory scheduling

Floorplanning

Notice that many real-world problems involve real-valued variables

A network of constraints

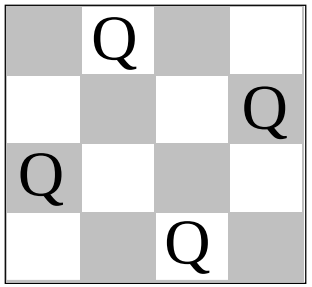


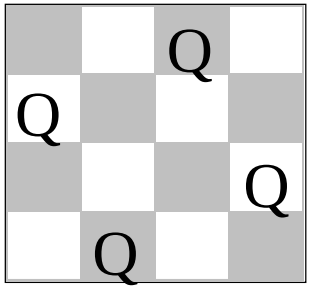
- **Variables**
  - $X_1,....,X_n$
- **Domains**
  - of discrete values: $D_1,....,D_n$
- **Binary constraints:**
  - R_{ij} which represent the list of allowed pairs of values, R_{ij} is a subset of the Cartesian product: $D_i x D_j$.
- **Constraint graph:**
  - A node for each variable and an arc for each constraint
- **Solution:**
  - An assignment of a value from its domain to each variable such that no constraint is violated.
- **A network of constraints represents the relation of all solutions.**

$$sol = \{(x_1,....,x_n) \mid (x_i, x_j) \in R_{ij}, x_i \in D_i, x_j \in D_j\}$$

Example 1: The 4-queen problem

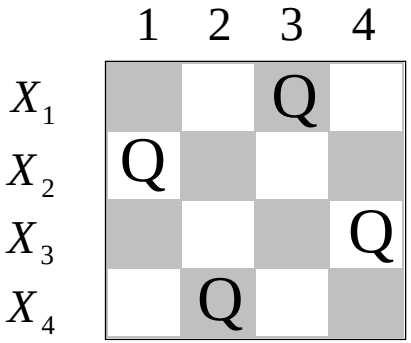
Place 4 Queens on a chess board of 4x4 such that no two queens reside in the same row, column or diagonal.





Standard CSP formulation of the problem:

- **Variables**: each row is a variable.

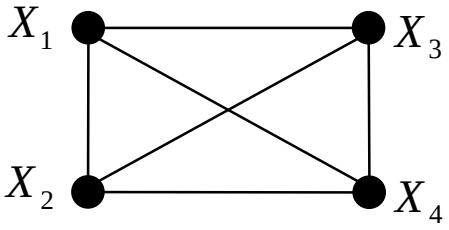


- **Domains**: $D_i = \{1,2,3,4\}.$

- **Constraints**: There are $\binom{4}{2}$ = 6 constraints involved:

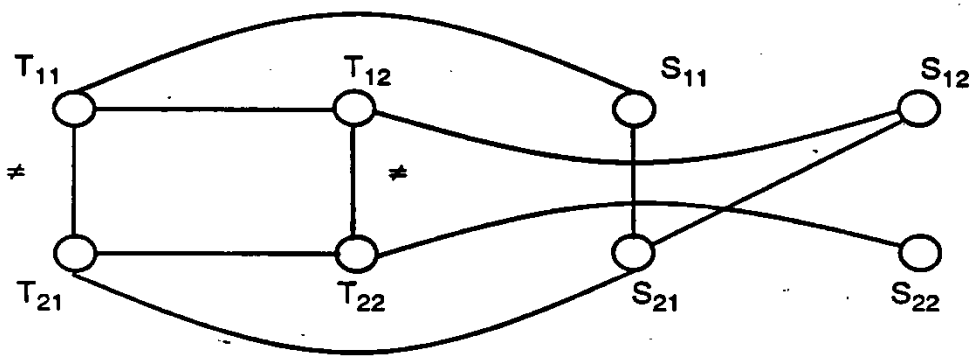
$$R_{12} = \{(1,3)(1,4)(2,4)(3,1)(4,1)(4,2)\}$$
$$R_{13} = \{(1,2)(1,4)(2,1)(2,3)(3,2)(3,4)(4,1)(4,3)\}$$
$$R_{14} = \{(1,2)(1,3)(2,1)(2,3)(2,4)(3,1)(3,2)(3,4)(4,2)(4,3)\}$$
$$R_{23} = \{(1,3)(1,4)(2,4)(3,1)(4,1)(4,2)\}$$
$$R_{24} = \{(1,2)(1,4)(2,1)(2,3)(3,2)(3,4)(4,1)(4,3)\}$$
$$R_{34} = \{(1,3)(1,4)(2,4)(3,1)(4,1)(4,2)\}$$

- **Constraint Graph** :



Class scheduling/Timetabling

- Teachers, Subjects, Classrooms, Time-slots.
- Constraints: - A teacher teaches a subset of subjects,
  - Subjects are taught at certain classrooms,
  - A teacher prefers teaching in the morning.
- Task: Assign a teacher and a subject to each class at each time slot, s.t. teachers' happiness is maximized.



T_{ij} - teacher at class C_i at time t_j $D(T_{ij})$ = teacher

S_{ij} - subject taught at class C_i at time t_j

Domain: subjects

Search vs. Inference

- **Search :**
  - In the space of partial variable-value assignments
  - Key idea : conditioning
- **Inference :**
  - Derive new information implied by values/constraints
  - Key idea : local consistency

Standard search formulation (incremental)

Let's start with the straightforward, dumb approach, then fix it

States are defined by the values assigned so far

♢ Initial state: the empty assignment, $\{\}$

♢ Successor function: assign a value to an unassigned variable
that does not conflict with current assignment.
$\Rightarrow$ fail if no legal assignments (not fixable!)

♢ Goal test: the current assignment is complete

1) This is the same for all CSPs!
2) Every solution appears at depth n with n variables
$\Rightarrow$ use depth-first search
3) Path is irrelevant, so can also use complete-state formulation
4) $b = (n - \ell)d$ at depth ℓ, hence $n!d^n$ leaves!!!!

Backtracking search

Variable assignments are commutative, i.e.,

$[WA = red \text{ then } NT = green]$ same as $[NT = green \text{ then } WA = red]$

Only need to consider assignments to a single variable at each node

$\Rightarrow$ $b = d$ and there are d^n leaves

Depth-first search for CSPs with single-variable assignments
is called backtracking search

Backtracking search is the basic uninformed algorithm for CSPs

Can solve n-queens for $n \approx 25$

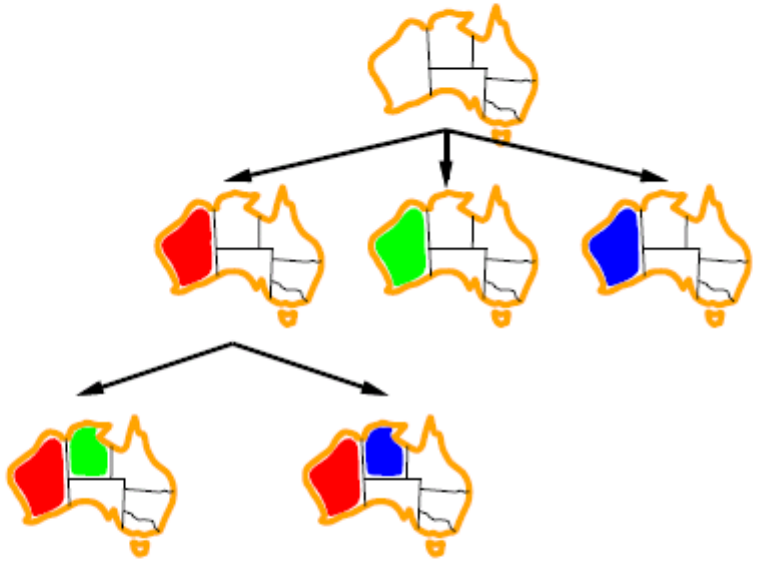
The search space

- **Definition: given an ordering of the variables** $X_1,....,X_n$
  - **a state:**
    - is an assignment to a subset of variables that is consistent.
  - **Operators:**
    - add an assignment to the next variable that does not violate any constraint.
  - **Goal state**:
    - a consistent assignment to **all** the variables.

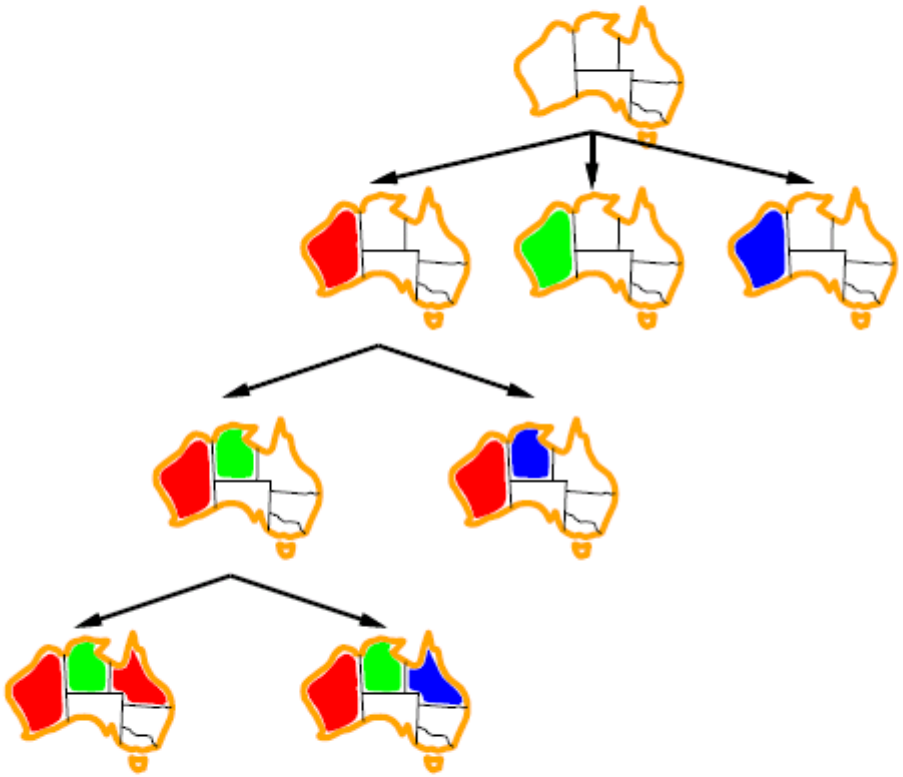
Backtracking example

Backtracking example

Backtracking example



Backtracking example



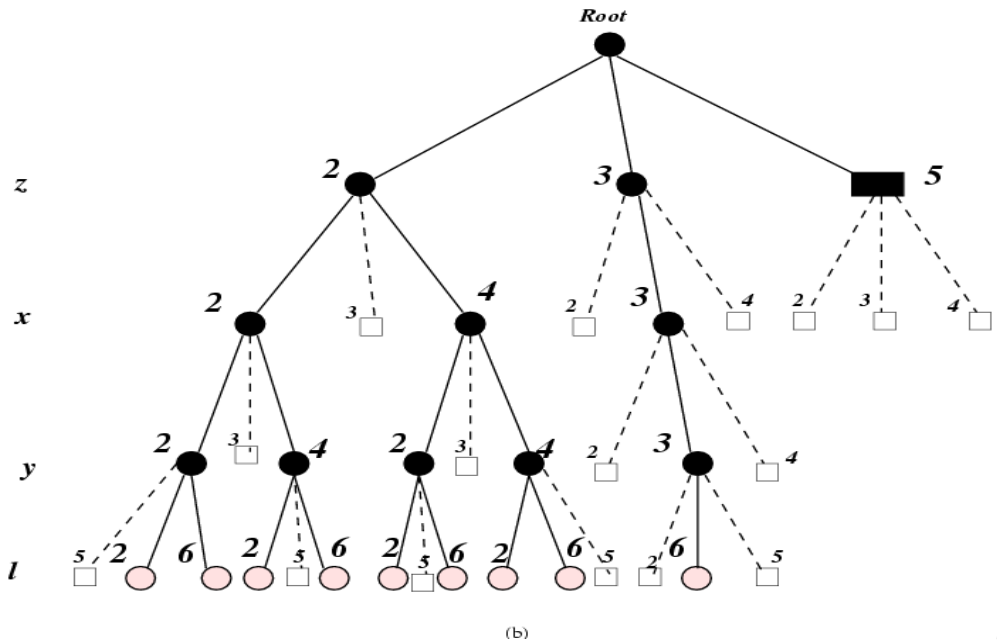
Backtracking

```
procedure BACKTRACKING
Input: A constraint network P = (X, D, C).
Output: Either a solution, or notification that the network is inconsistent.

    i ← 1                              (initialize variable counter)
    D'_i ← D_i                         (copy domain)
    while 1 ≤ i ≤ n
       instantiate x_i ← SELECTVALUE
       if x_i is null                  (no value was returned)
          i ← i − 1                    (backtrack)
       else
          i ← i + 1                    (step forward)
          D'_i ← D_i
    end while
    if i = 0
       return "inconsistent"
    else
       return instantiated values of {x_1, ..., x_n}
end procedure

subprocedure SELECTVALUE   (return a value in D'_i consistent with ā_{i−1})

    while D'_i is not empty
       select an arbitrary element a ∈ D'_i, and remove a from D'_i
       if CONSISTENT(ā_{i−1}, x_i = a)
          return a
    end while
    return null                        (no consistent value)
end procedure
```

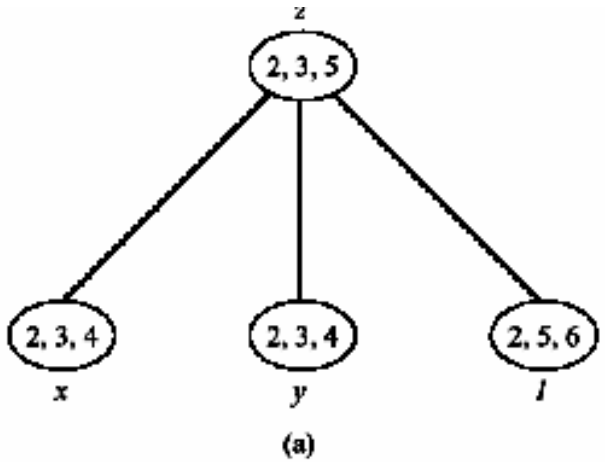
Figure 5.4: The backtracking algorithm.

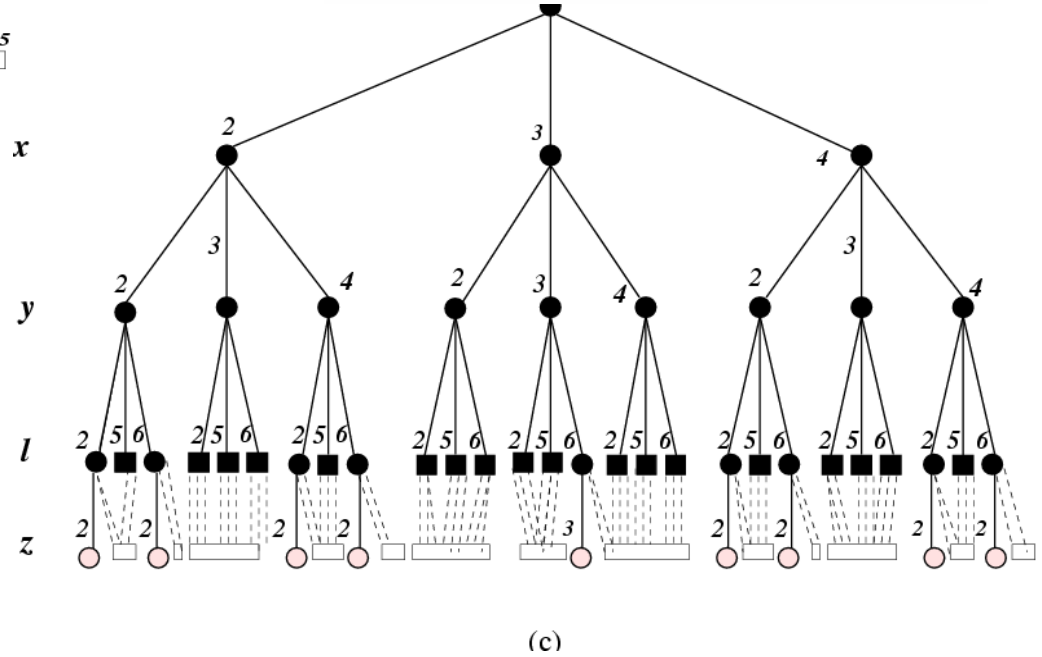
- **Complexity of extending a partial solution:**
  - **Complexity of consistent: O(e log t), t bounds #tuples, e bounds #constraints**
  - **Complexity of selectvalue: O(e k log t), k bounds domain size**

The effect of variable ordering



z divides x, y and t





A coloring problem

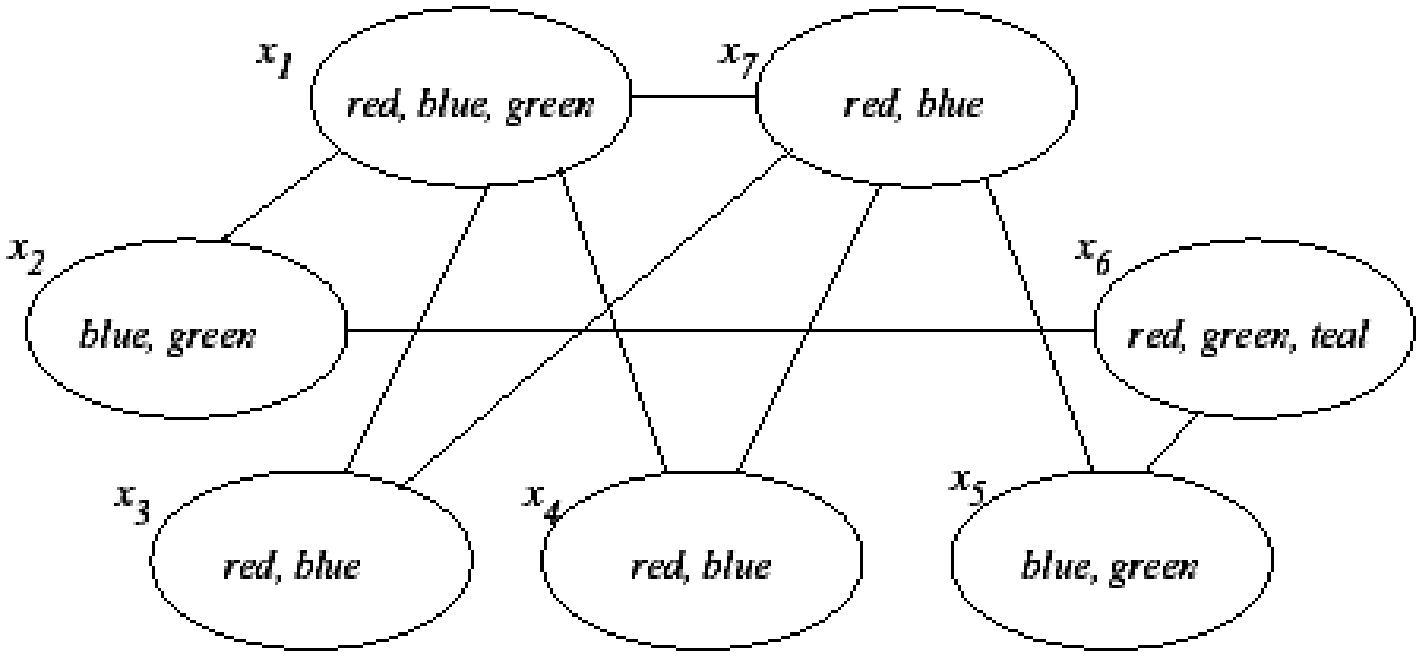
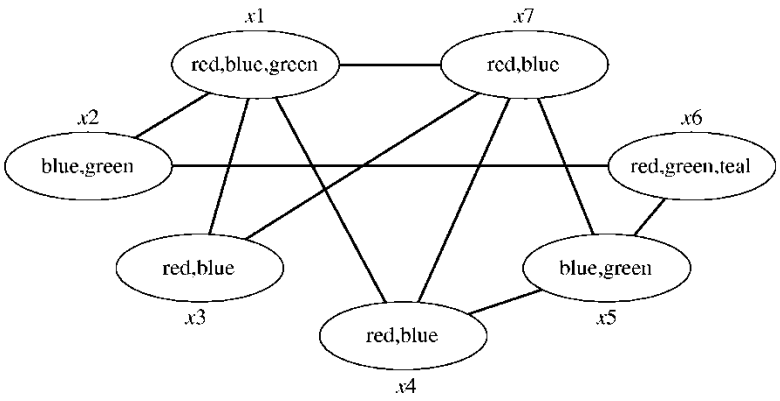
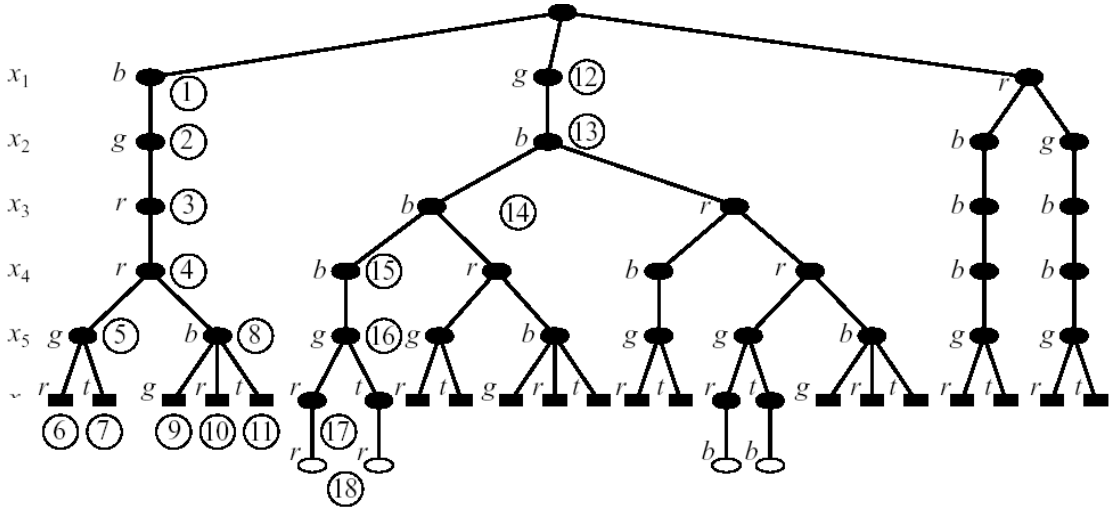


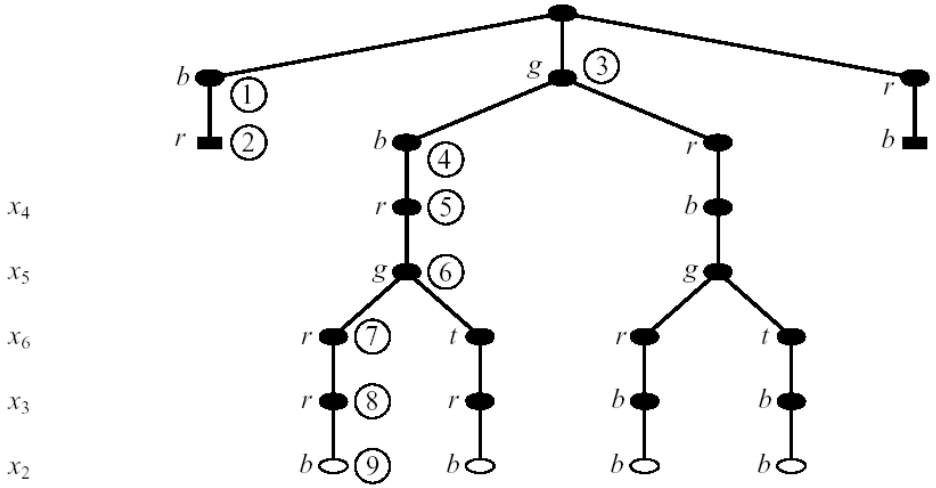
Figure 5.3: A coloring problem with variables $(x_1, x_2, \ldots, x_7)$. The domain of each variable is written inside the corresponding node. Each arc represents the constraint that the two variables it connects must be assigned different colors.

Backtracking Search for a Solution



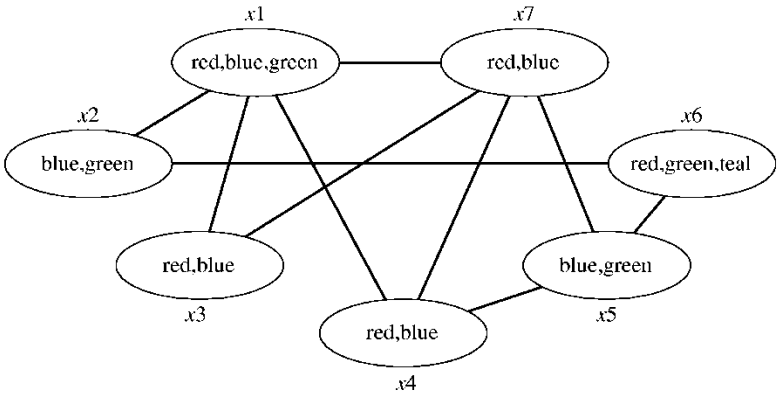


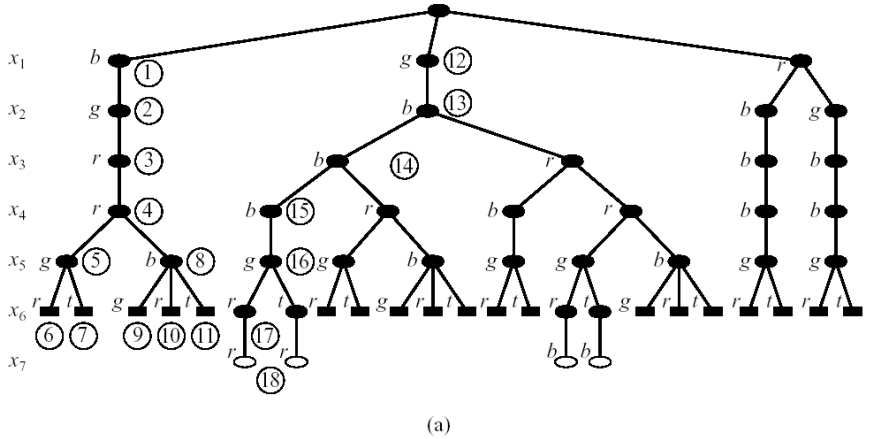
(a)



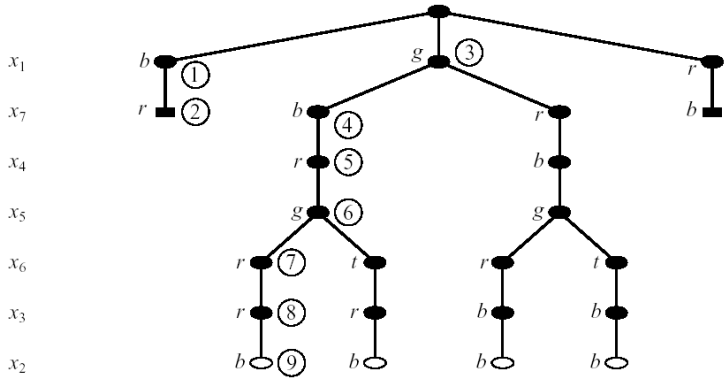
(b)

Backtracking Search for a Solution



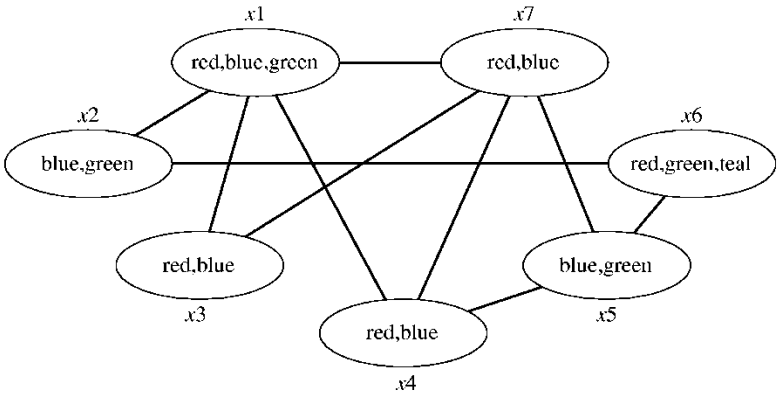


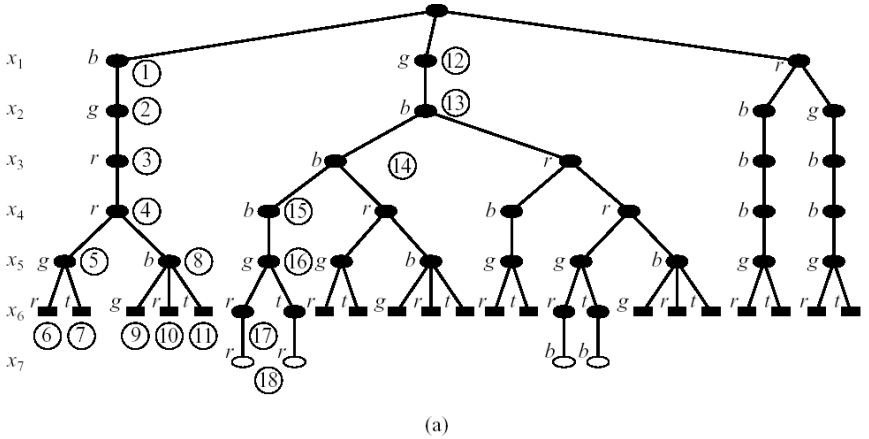
(a)



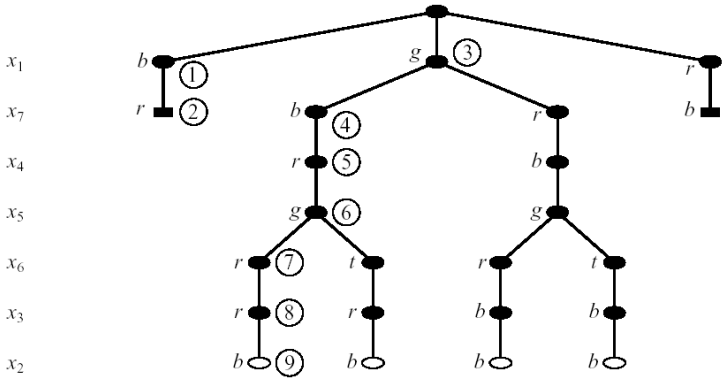
(b)

Backtracking Search for All Solutions





(a)

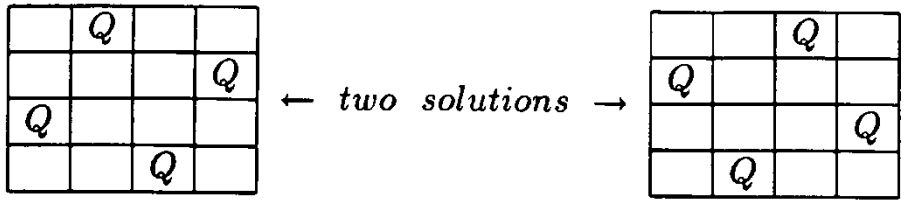


(b)

Summary so far

- **Constraint Satisfaction Problems**
  - Variables
  - Domains
  - Constraints (as a relation)
- **Solution**
  - Assignment of values to variables such that all constraints are satisfied
- **Constraint Graph**
- **Backtracking Search (DFS)**
  - Given a variable ordering
  - Given a current partial variable ordering, extend it to the next variable along the given ordering, such that the value of the newly assigned variable is consistent with all previously assigned variables
  - Backtrack if a dead end
  - Ideal : backtrack-free search
- **Impact of variable order on search space size**

The Minimal network: Example: the 4-queen problem



	1	2	3	4
$X_1 \rightarrow$		Q		
$X_2 \rightarrow$				Q
$X_3 \rightarrow$	Q			
$X_4 \rightarrow$			Q	

$$R_{12} = \{ (1,3)\ (1,4)\ (2,4)\ (3,1)\ (4,1)\ (4,2) \}$$
$$R_{13} = \{ (1,2)\ (1,4)\ (2,1)\ (2,3)\ (3,2)\ (3,4)\ (4,1)\ (4,3) \}$$
$$R_{14} = \{ (1,2)\ (1,3)\ (2,1)\ (2,3)\ (2,4)\ (3,1)\ (3,2)\ (3,4)\ (4,2)\ (4,3) \}$$
$$R_{23} = \{ (1,3)\ (1,4)\ (2,4)\ (3,1)\ (4,1)\ (4,2) \}$$
$$R_{24} = \{ (1,2)\ (1,4)\ (2,1)\ (2,3)\ (3,2)\ (3,4)\ (4,1)\ (4,3) \}$$
$$R_{34} = \{ (1,3)\ (1,4)\ (2,4)\ (3,1)\ (4,1)\ (4,2) \}$$

$$\textit{the solution}\ \ \rho = \left(\begin{array}{cccc} X_1 & X_2 & X_3 & X_4 \\ \hline 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \end{array}\right)$$

$$M_\rho = proj(\rho) = \left\{ \begin{array}{l} M_{12} = \{ (2,4)\ (3,1) \} \\ M_{13} = \{ (2,1)\ (3,4) \} \\ M_{14} = \{ (2,3)\ (3,2) \} \\ M_{23} = \{ (1,4)\ (4,1) \} \\ M_{24} = \{ (1,2)\ (4,3) \} \\ M_{34} = \{ (1,3)\ (4,2) \} \end{array} \right\}$$

Inference (Approximation) algorithms

- **Arc-consistency (Waltz, 1972)**
- **Path-consistency (Montanari 1974, Mackworth 1977)**
- **k-consistency (Freuder 1982)**
- **Key Idea :**
  - Transform the network into smaller and smaller (but equivalent) networks by tightening the domains/constraints.

Arc-consistency

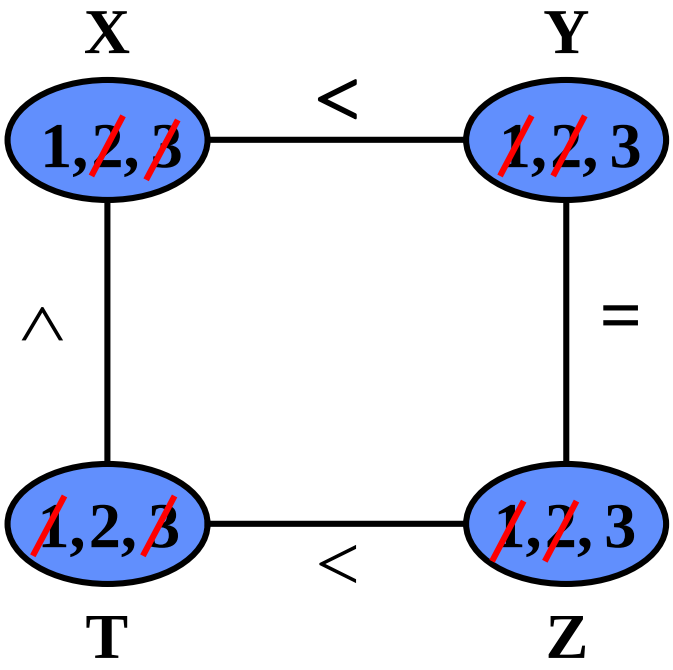
$1 \leq X, Y, Z, T \leq 3$

$X < Y$

$Y = Z$

$T < Z$

$X < T$



Arc-consistency

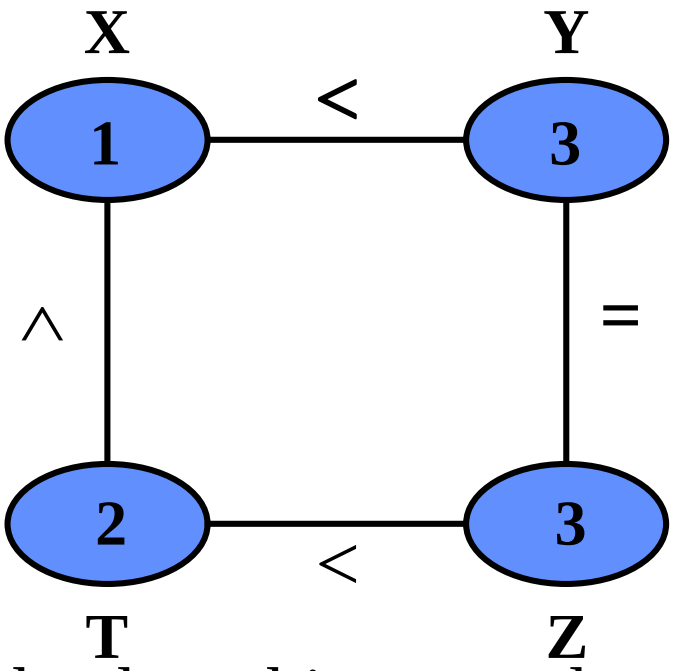
$1 \leq X, Y, Z, T \leq 3$

$X < Y$

$Y = Z$

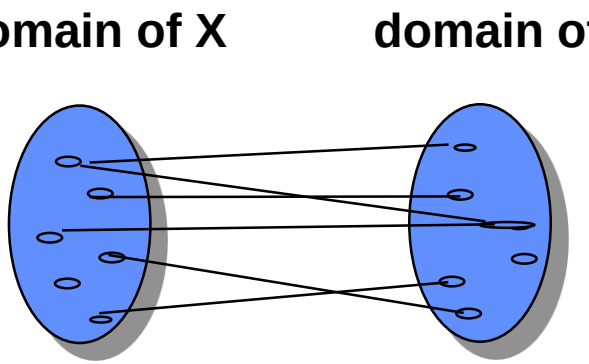
$T < Z$

$X < T$



- Incorporated into backtracking search
- Constraint programming languages powerful approach for modeling and solving combinatorial optimization problems.

Arc-consistency algorithm



Arc (X_i, X_j) is arc-consistent if for any value of X_i there exist a matching value of X_i

Algorithm Revise (X_i, X_j) makes an arc consistent

Begin

1. For each a in D_i if there is no value b in D_j that is consistent with a then delete a from the D_i.

End.

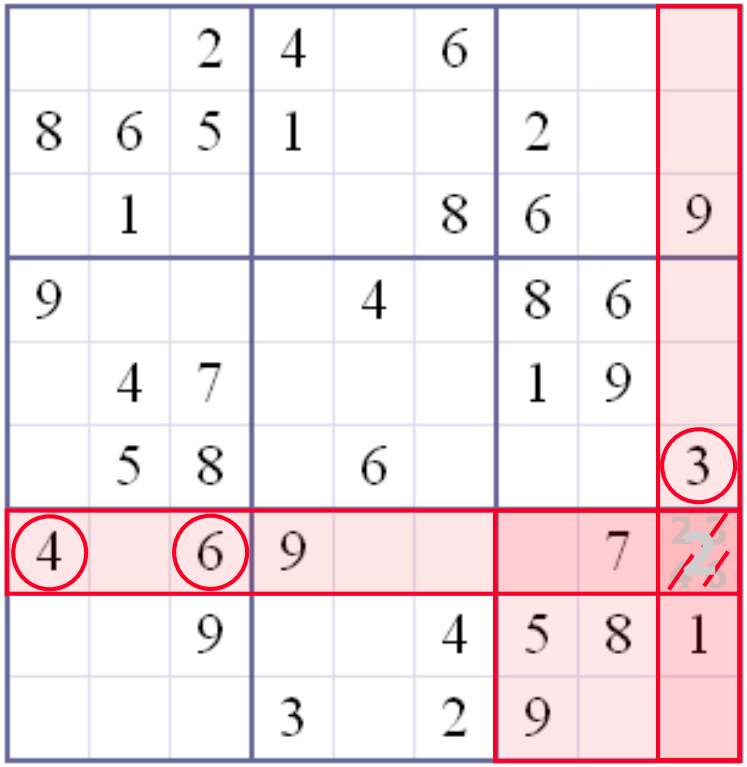
Revise is $O(k^2)$, *k* is the number of values in each domain.

Algorithm AC-3

- **Begin**
  - 1. Q <--- put all arcs in the queue in both directions
  - 2. While Q is not empty do,
  - 3. Select and delete an arc (X_i,X_j) from the queue Q
    - 4. Revise (X_i,X_j)
    - 5. If Revise changes (reduces) the domain of X_i then add to the queue all arcs that go to X_i ($(X_l,X_i$ for all l except i).
  - 6. end-while
- **End**
- **Complexity:**
  - Processing an arc requires O(k^2) steps
  - There is e edges
  - The number of times each arc can be processed is 2·k
  - Total complexity is $O(ek^3)$

Sudoku

Constraint propagation

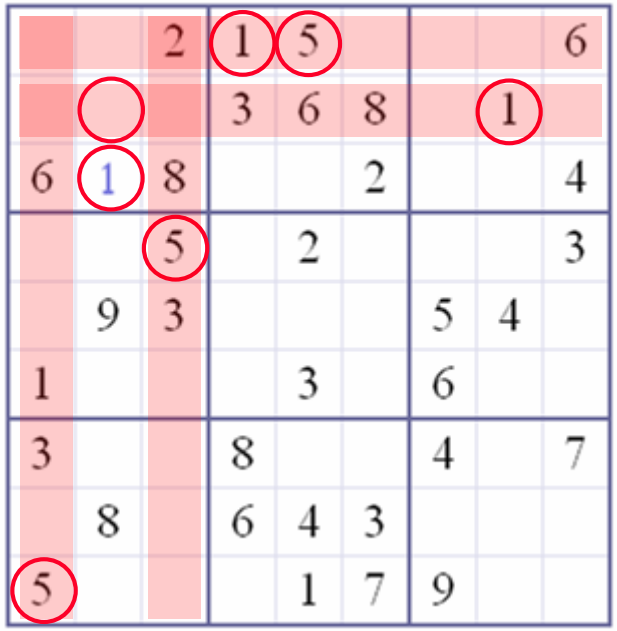


- **Variables: 81 slots**
- **Domains = {1,2,3,4,5,6,7,8,9}**
- **Constraints:**
  - **27 not-equal**

Each row, column and major block must be all different

"Well posed" if it has unique solution: 27 constraints

Sudoku



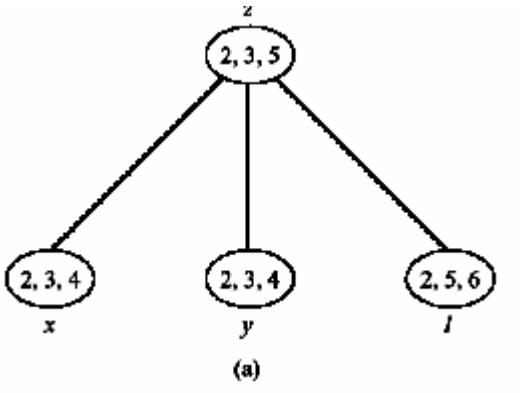
Path-consistency or
3-consistency

4-consistency and
i-consistency in geeral

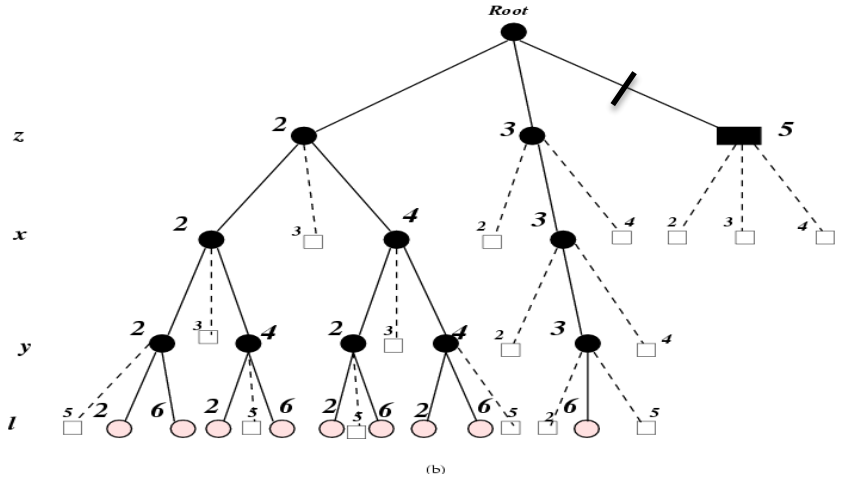
Each row, column and major block must be all different

“Well posed” if it has unique solution

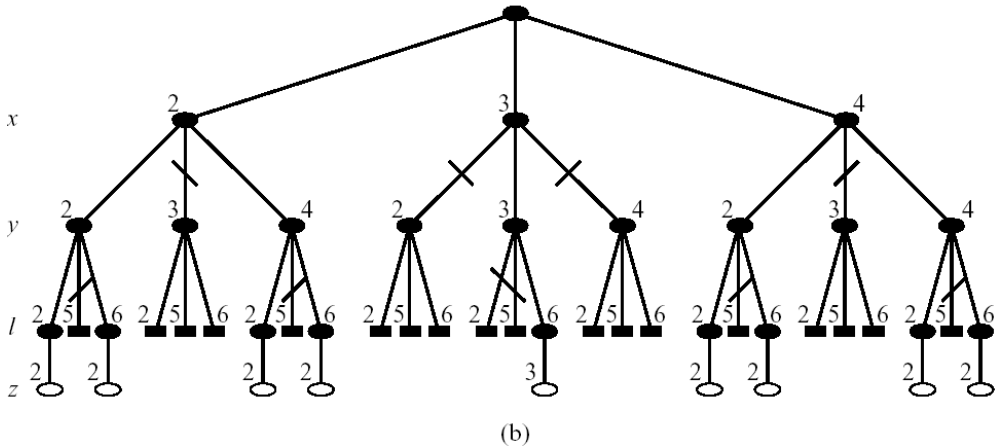
The Effect of Consistency Level



- **After arc(2)-consistency z=5 and l=5 are removed**



- **After path(3)-consistency**
  - R'_zx
  - R'_zy
  - R'_zl
  - R'_xy
  - R'_xl
  - R'_yl



Tighter networks yield smaller search spaces

Improving Backtracking O(exp(n))

- **Before search: (reducing the search space)**
  - Arc-consistency, path-consistency, k-consistency
  - Variable ordering (fixed)
- **During search:**
  - **Look-ahead schemes**:
    - Variable ordering (*Choose the most constraining variable)*
    - Value ordering (*Choose the least restricting value*)
  - **Look-back schemes**:
    - Backjumping
    - Constraint recording
    - Dependency-directed backtracking

Improving backtracking efficiency

General-purpose methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?

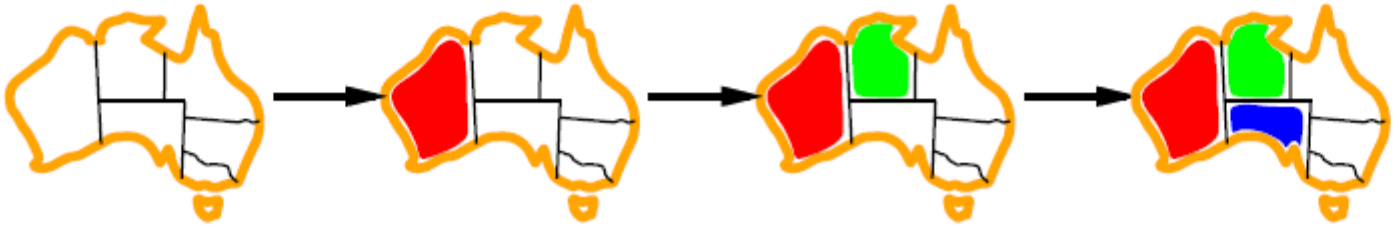
Look-ahead: Variable and value orderings

- **Intuition:**
  - Choose a variable that will detect failures early
  - Choose a value least likely to yield a dead-end
  - Approach: apply propagation at each node in the search tree
- **Forward-checking**
  - Check each unassigned variable separately
- **Maintaining arc-consistency (MAC)**
  - Apply full arc-consistency

Most constrained variable

Most constrained variable:

choose the variable with the fewest legal values

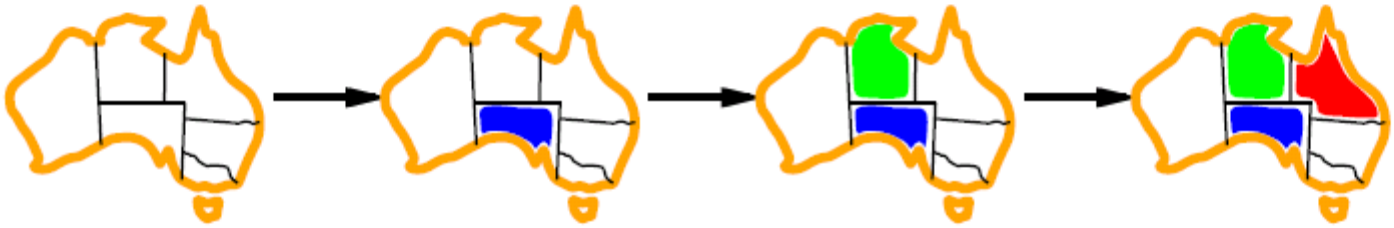


Most constraining variable

Tie-breaker among most constrained variables

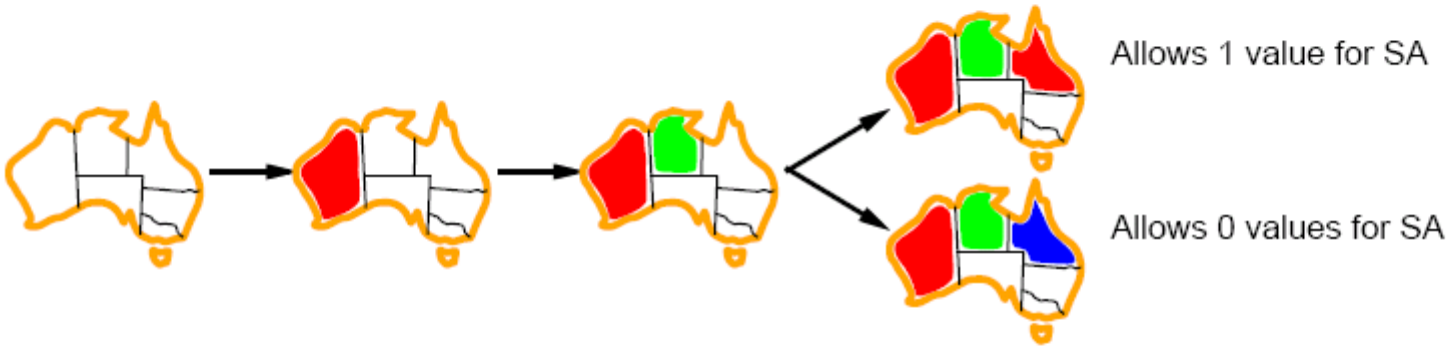
Most constraining variable:

choose the variable with the most constraints on remaining variables



Least constraining value

Given a variable, choose the least constraining value:
the one that rules out the fewest values in the remaining variables



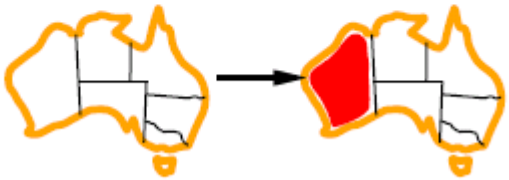
Combining these heuristics makes 1000 queens feasible

Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

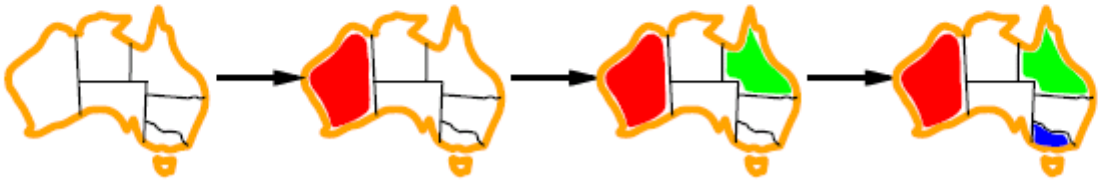


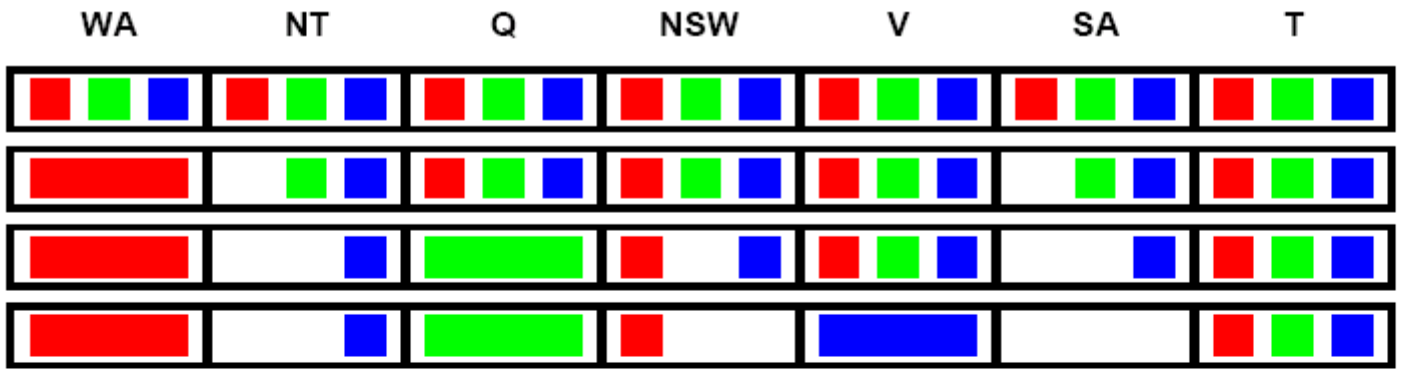
Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values





Constraint propagation

Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

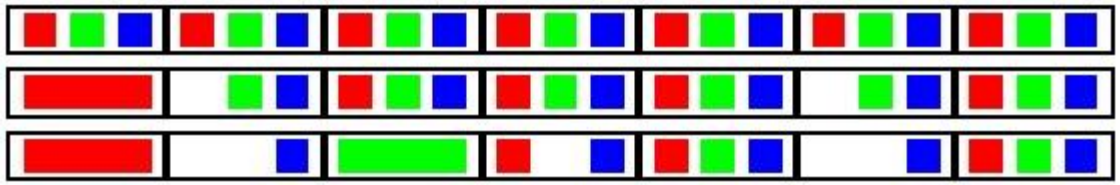
NT and SA cannot both be blue!

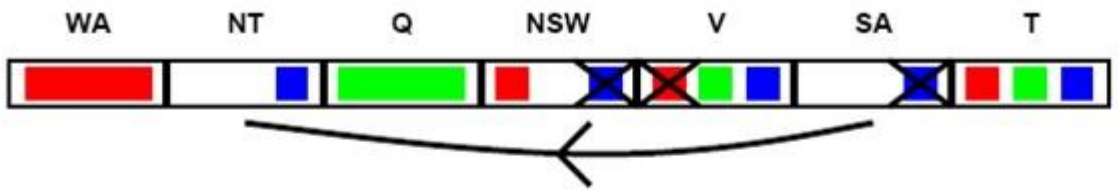
Constraint propagation repeatedly enforces constraints locally

Arc consistency

Simplest form of propagation makes each arc consistent

$X \rightarrow Y$ is consistent iff
for *every* value x of X there is *some* allowed y





If X loses a value, neighbors of X need to be rechecked

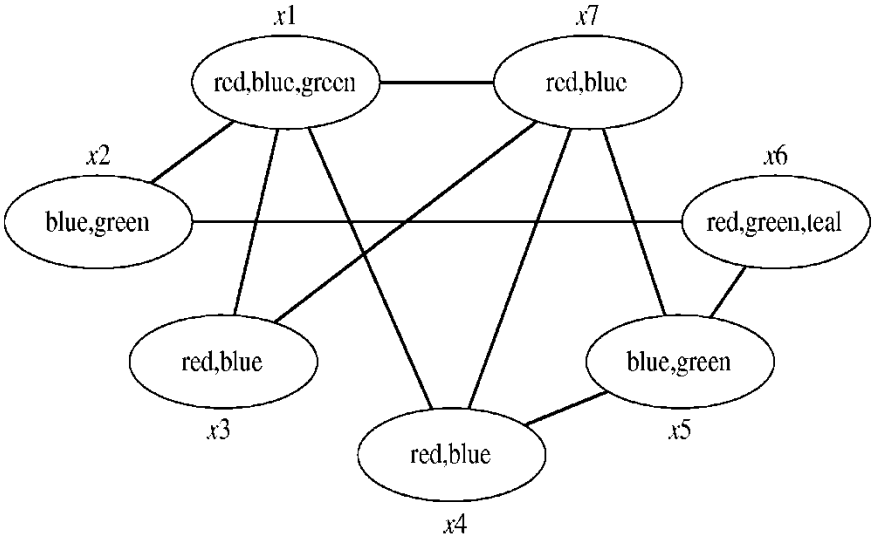
Arc consistency detects failure earlier than forward checking

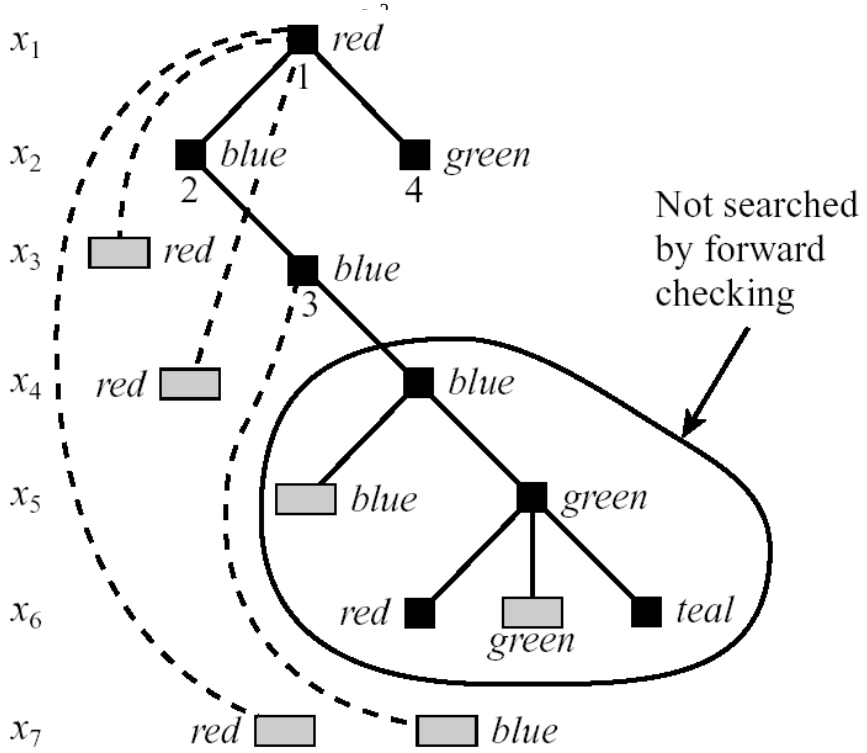
Can be run as a preprocessor or after each assignment

Summary so far

- **Local consistency**
  - Arc(2)-consistency (AC-3) : virtually always used
  - Path(3)-consistency : usually not used
  - K-consistency
  - Key ideas :
    - Tighten domains/constraints
    - Iteratively propagate effects
- **Effect of local consistency of backtracking search space size**
- **Improving BT search**
  - Variable ordering : pick first variable most likely to fail
    - MRV heuristic
  - Value ordering : pick value most likely to succeed
    - Least constraining value heuristic
- **Lookahead schemes**
  - Forward checking
  - Maintaining arc-consistency

Forward-checking on Graph-coloring





FW overhead: $O(ek^2)$

MAC overhead: $O(ek^3)$

Algorithm DVO (DVFC)

procedure DVFC
Input: A constraint network $\mathcal{R} = (X, D, C)$
Output: Either a solution, or notification that the network is inconsistent.

```
D'_i ← D_i for 1 ≤ i ≤ n        (copy all domains)
i ← 1                          (initialize variable counter)
        s = min_{i<j≤n} |D'_j|  (find future var with smallest domain)
        x_{i+1} ← x_s          (rearrange variables so that x_s follows x_i)
while 1 ≤ i ≤ n
   instantiate x_i ← SELECTVALUE-FORWARD-CHECKING
   if x_i is null              (no value was returned)
      reset each D' set to its value before x_i was last instantiated
      i ← i − 1                (backtrack)
   else
      if i < n
      i ← i + 1                (step forward to x_s)
         s = min_{i<j≤n} |D'_j| (find future var with smallest domain)
         x_{i+1} ← x_s         (rearrange variables so that x_s follows x_i)
      i ← i + 1                (step forward to x_s)
end while
if i = 0
   return "inconsistent"
else
   return instantiated values of {x_1, ..., x_n}
end procedure
```

Figure 5.12: The DVFC algorithm. It uses the SELECTVALUE-FORWARD-CHECKING sub-procedure given in Fig. 5.8.

Propositional Satisfiability

Example: party problem

- **If Alex goes, then Becky goes:** $\mathbf{A} \rightarrow \mathbf{B} \quad (\text{or}, \neg\mathbf{A} \vee \mathbf{B})$
- **If Chris goes, then Alex goes:** $\mathbf{C} \rightarrow \mathbf{A} \quad (\text{or}, \neg\mathbf{C} \vee \mathbf{A})$
- Query:

  Is it possible that Chris goes to the party but Becky does not?

$\Updownarrow$

Is *propositional theory*

$\varphi = \{\neg\mathbf{A} \vee \mathbf{B},\ \neg\mathbf{C} \vee \mathbf{A},\ \neg\mathbf{B},\ \mathbf{C}\}$ satisfiable?

Unit Propagation

- **Arc-consistency for CNFs.**
- **Involve a single clause and a single literal**
- **Example:** $(\mathbf{A} \vee \neg \mathbf{B} \vee C), \mathrm{B} \quad \rightarrow \quad (\mathbf{A} \vee C)$

Look-ahead for SAT

(Davis-Putnam, Logeman and Laveland, 1962)

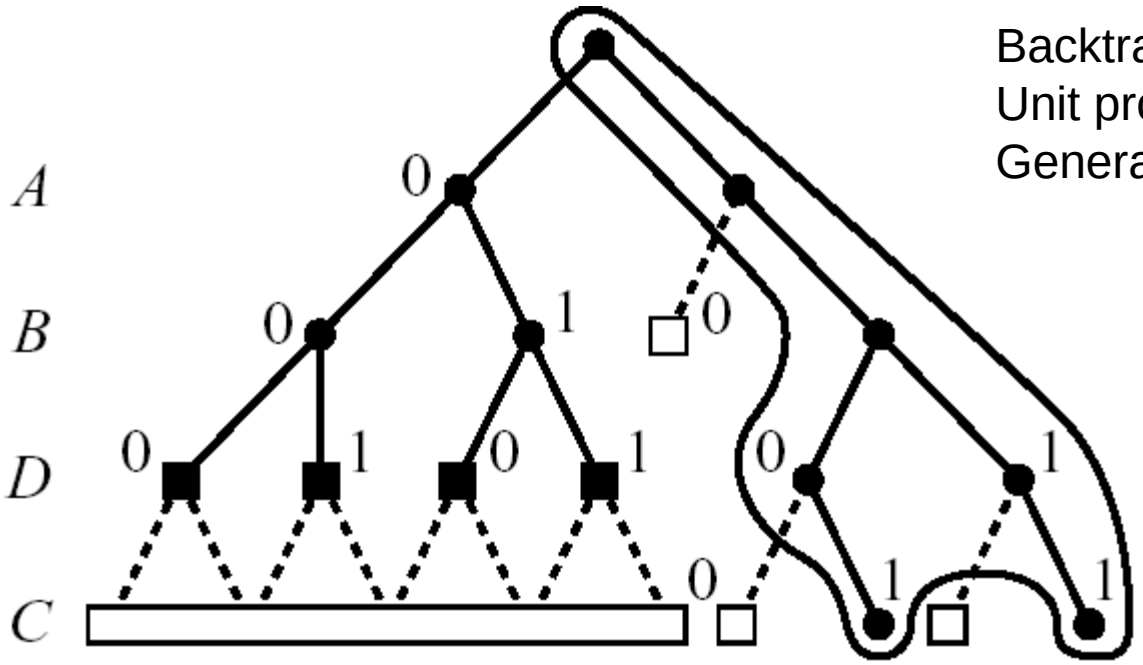
DPLL(φ)
Input: A cnf theory φ
Output: A decision of whether φ is satisfiable.
1. Unit_propagate(φ);
2. If the empty clause is generated, return(*false*);
3. Else, if all variables are assigned, return(*true*);
4. Else
5. $\quad Q$ = some unassigned variable;
6. $\quad$ return(**DPLL**($\varphi \wedge Q$) $\vee$
$\qquad\qquad$ **DPLL**($\varphi \wedge \neg Q$))

Figure 5.13: The DPLL Procedure

Look-ahead for SAT: DPLL

(Davis-Putnam, Logeman and Laveland, 1962)

example: (¬ A V B) ∧ (¬ C V A) ∧ (A V B V D) ∧ (C)



Backtracking look-ahead with
Unit propagation=
Generalized arc-consistency

Only enclosed area will be explored with unit-propagation

Look-back:
Backjumping / Learning

- **Backjumping:**
  - In deadends, go back to the most recent culprit.
- **Learning:**
  - constraint-recording, no-good recording.
  - good-recording

Backjumping

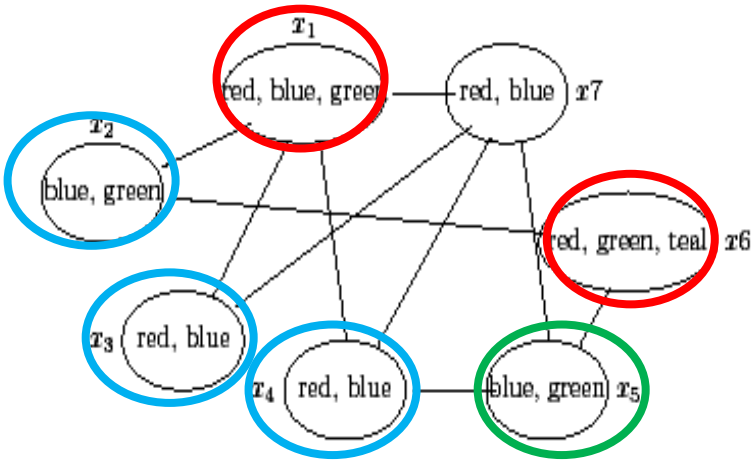
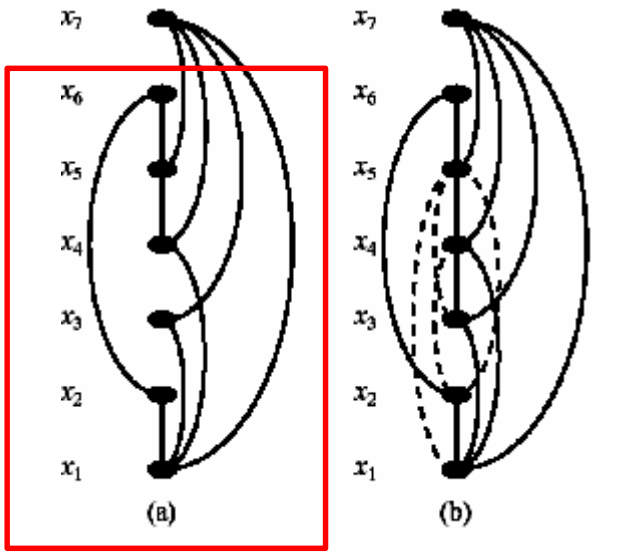


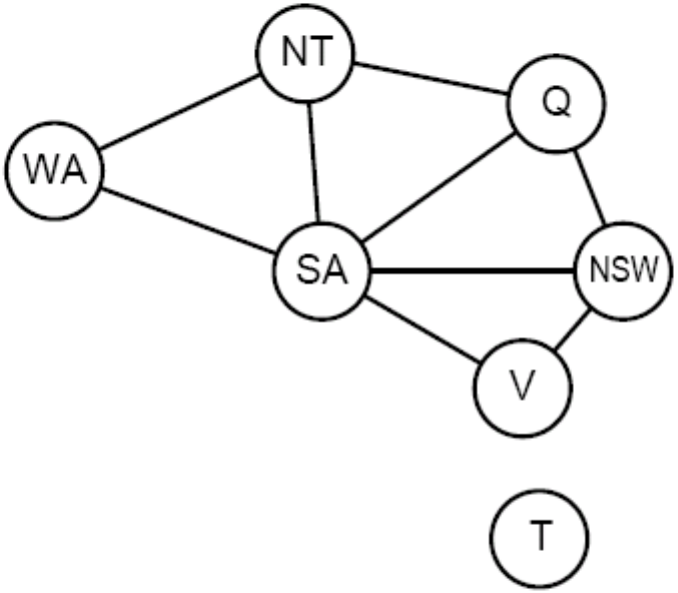
Figure 6.1: A modified coloring problem.

?



- **(X1=r,x2=b,x3=b,x4=b,x5=g,x6=r,x7={r,b})**
- **(r,b,b,b,g,r)** conflict set **of x7**
  - Conflict set : a partial assignment that cannot be extended to the next variable
- **(r,-,b,b,g,-) c.s. of x7**
- **(r,-,b,-,-,-,-)** minimal conflict-set
- Leaf deadend**: (r,b,b,b,g,r)**
  - No-good : a partial assignment that cannot be extended to a solution
  - No-good = internal deadend
- **Every conflict-set is a** no-good
  - **But not other way around (x1=r is a** no-good**)**

Problem structure



Tasmania and mainland are independent subproblems

Identifiable as connected components of constraint graph

Problem structure contd.

Suppose each subproblem has c variables out of n total

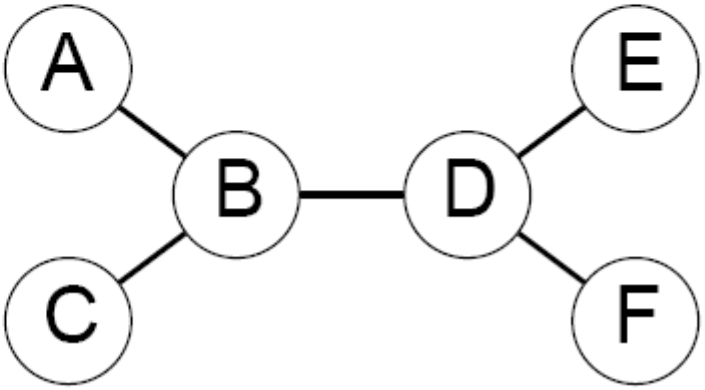
Worst-case solution cost is $n/c \cdot d^c$, *linear* in n

E.g., $n = 80$, $d = 2$, $c = 20$

$2^{80} =$ 4 billion years at 10 million nodes/sec

$4 \cdot 2^{20} =$ 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs



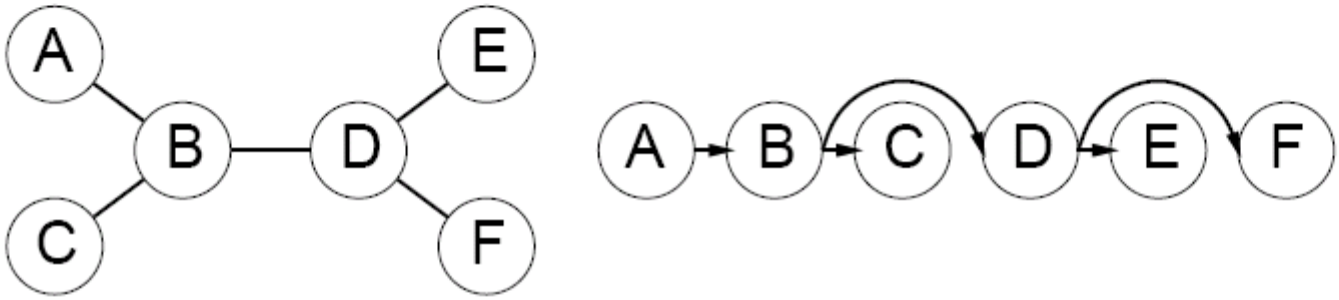
Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning:
an important example of the relation between syntactic restrictions
and the complexity of reasoning.

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering



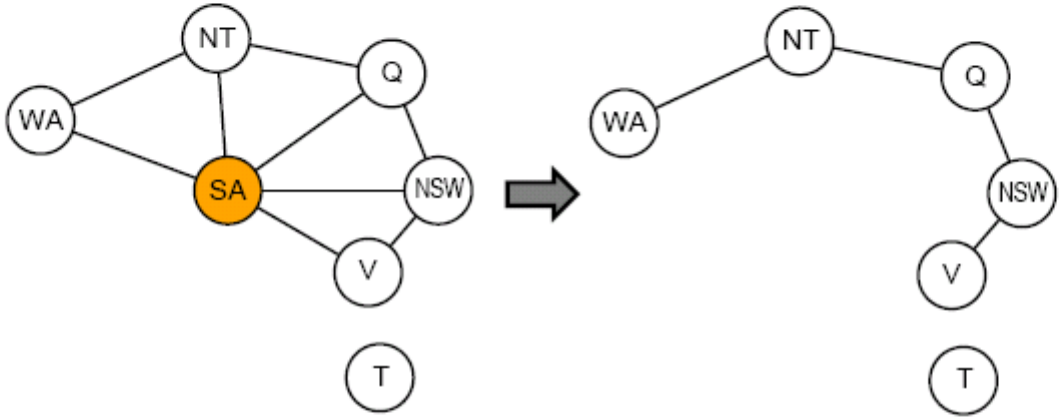
2. For j from n down to 2, apply $\textsc{RemoveInconsistent}(Parent(X_j), X_j)$

3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

4. Arc-consistency in tree-structured CSPs makes search backtrack-free

Nearly tree-structured CSPs

Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

Cutset size c $\Rightarrow$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

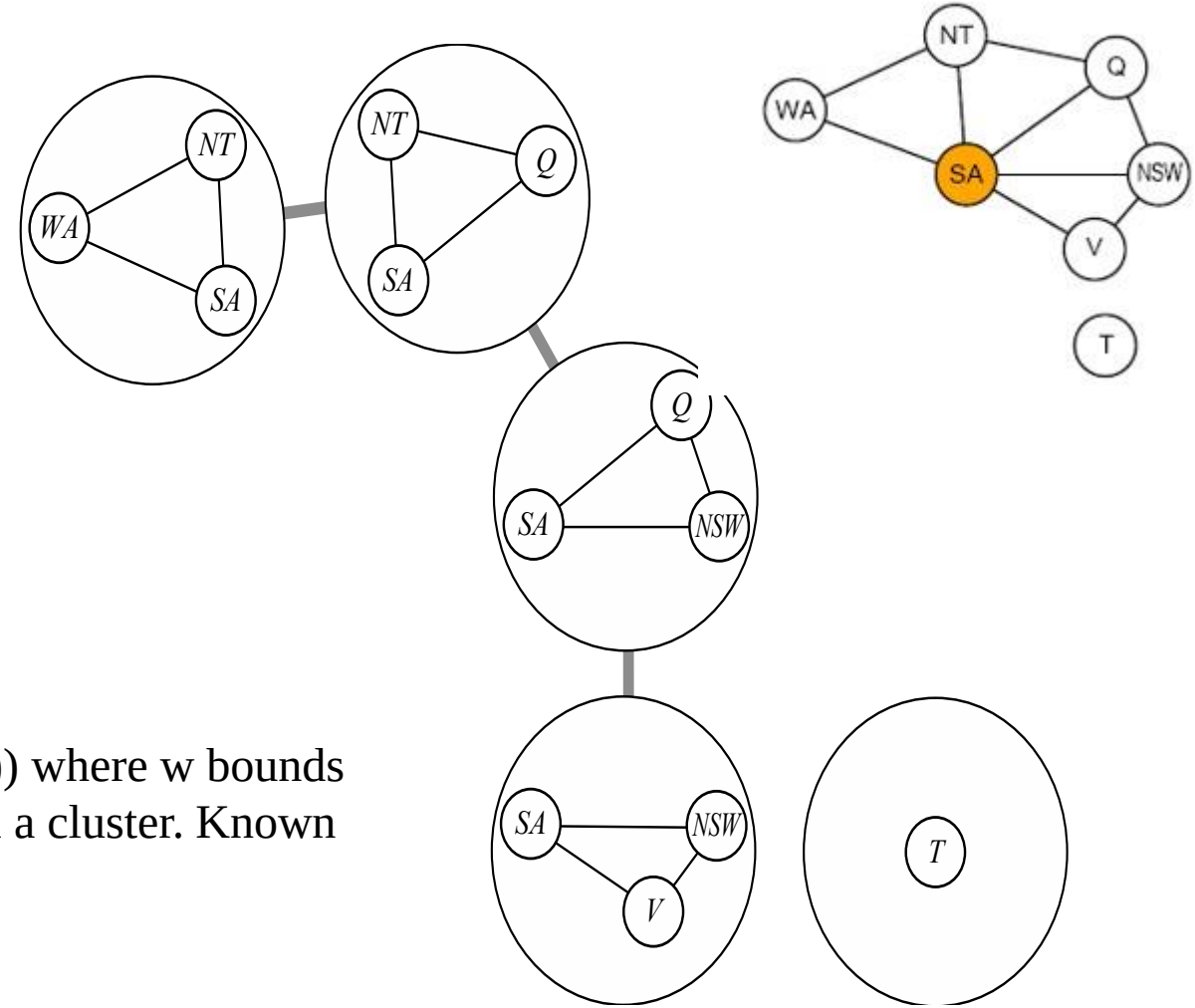
The cycle-cutset method

- **An instantiation can be viewed as blocking cycles in the graph**
- **Given an instantiation to a set of variables that cut all cycles (a cycle-cutset) the rest of the problem can be solved in linear time by a tree algorithm.**

- **Complexity (n number of variables, k the domain size and C the cycle-cutset size):**

$$O(nk^C k^2)$$

Tree Decomposition



Complexity is O(n exp(w)) where w bounds the number of variables in a cluster. Known as the treewidth.

Local Search for CSPs

- **Local Search**
  - Works with complete variable assignments (called a state)
  - Has a cost function associated with each state
    - E.g. min-conflicts – number of constraints violated
  - Considers only immediate neighborhood for next state
    - E.g. change the value of one variable
- **Operators**
  - Greedy heuristic : pick a value for a variable that leads to greatest reduction in total cost – can get stuck in local optima
  - Random moves : occasionally make a random move
  - Sideways moves, re-starts, constraint re-weighting, …
- **Incomplete**
- **Often fast**
  - orders of magnitude faster than systematic e.g. DFS

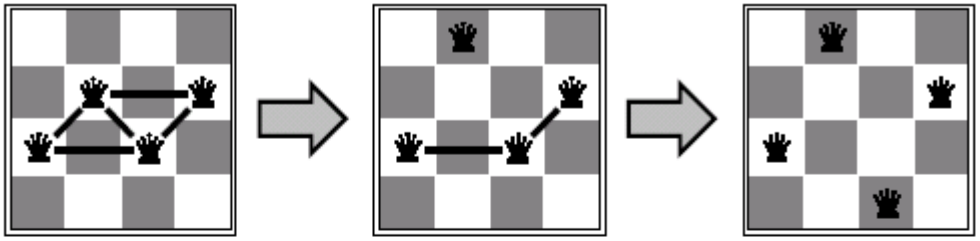
Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)

Operators: move queen in column

Goal test: no attacks

Evaluation: $h(n) =$ number of attacks



GSAT – local search for SAT

(Selman, Levesque and Mitchell, 1992)

```
1.      For i=1 to MaxTries
2.          Select a random assignment A
3.          For j=1 to MaxFlips
4.              if    A satisfies all constraints, return A
5.              else   flip a variable to maximize the score
6.                  (number of satisfied constraints; if no variable
7.                  assignment increases the score, flip at random)
8.          end
9.      end
```

Greatly improves hill-climbing by adding restarts and sideway moves

WalkSAT

(Selman, Kautz and Cohen, 1994)

Adds random walk to GSAT:

With probability p

random walk – flip a variable in some unsatisfied constraint

With probability 1-p

perform a hill-climbing step

Randomized hill-climbing often solves large and hard satisfiable problems

More Stochastic Search: Simulated Annealing, Constraint Reweighting

- **Simulated annealing:**
  - A method for overcoming local minimas
  - Allows bad moves with some probability:
    - With some probability related to a temperature parameter T the next move is picked randomly.
  - Theoretically, with a slow enough cooling schedule, this algorithm will find the optimal solution. But so will searching randomly.

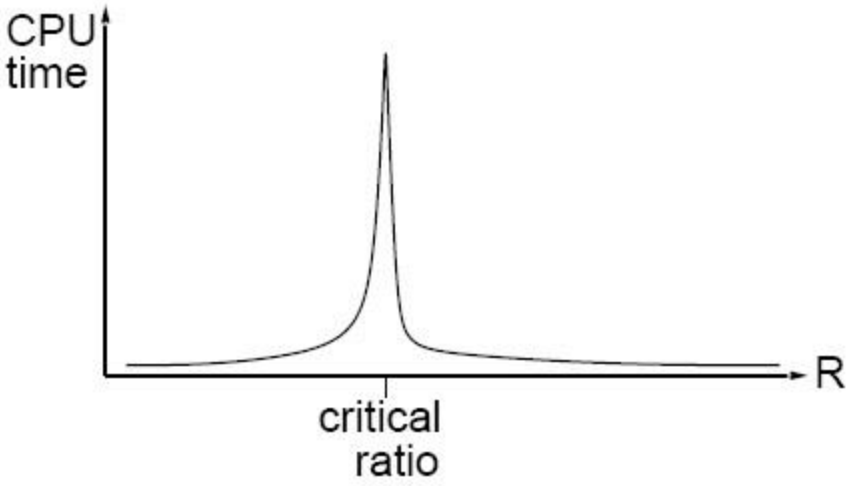
- **Breakout method (Morris, 1990): adjust the weights of the violated constraints**

Problem Hardness

Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., $n = 10{,}000{,}000$)

The same appears to be true for any randomly-generated CSP
except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

CSPs are a special kind of problem:
- states defined by values of a fixed set of variables
- goal test defined by *constraints* on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g., arc consistency) does additional work
to constrain values and detect inconsistencies

The CSP representation allows analysis of problem structure

Tree-structured CSPs can be solved in linear time

Iterative min-conflicts is usually effective in practice