## Chapter 6
## Randomized Block Design
## Two Factor ANOVA
## Interaction in ANOVA

## Two factor (two-way) ANOVA

Two-factor ANOVA is used when:

- Y is a quantitative response variable
- There are two categorical explanatory variables, called Factors:
  - Factor A has K levels, $k = 1, \dots, K$
  - Factor B has J levels, $j = 1, \dots, J$
- The combination of level $k$ for A and level $j$ for B has sample size $n_{kj}$ but if all equal, just use $n$.
- Use N for overall sample size.

## Special case: Using "BLOCKS"

Definition: A <u>block</u> is a group of similar units, or the same unit measured multiple times.

Blocks are used to reduce known sources of variability, by comparing levels of a factor <u>within</u> blocks.

Examples (explained in detail in class):

- Factor = 3 methods of reducing blood pressure; Blocks defined using initial blood pressure.
- Factor = 4 methods for enhancing memory; Blocks defined by age.
- Factor = Impairment while driving (alcohol, marijuana, no sleep, control); Blocks = individuals.

## Simple Block Design, all $n_{kj} = 1$

A <u>simple block design</u> has two factors with:

- Exactly one data value (observation) in each combination of the factors.
- Factor A is factor of interest, called *treatment*
- Factor B, called *blocks*, used to control a known source of variability

Main interest is comparing levels of the *treatment.*

Notation: Factor A (Treatments) has $K$ levels
Factor B (Blocks) has $J$ levels
$\rightarrow N = KJ$ data values

## Example: Do Means Differ for 4 Exam Formats?

| | Adam | Brenda | Cathy | Dave | Emily | Mean |
|---|---|---|---|---|---|---|
| Exam #1: | 62 | 94 | 68 | 86 | 50 | 72 |
| Exam #2: | 87 | 95 | 93 | 97 | 63 | 87 |
| Exam #3: | 74 | 86 | 82 | 70 | 28 | 68 |
| Exam #4: | 77 | 89 | 73 | 79 | 47 | 73 |
| Mean | 75 | 91 | 79 | 83 | 47 | 75 |

Treatments: 4 different exam formats, Blocks: 5 different students

Question: Is there a difference in population means for the 4 exams?

Use students as *blocks* because we know student abilities differ. Controls for that *known* source of variability.

## Two-way ANOVA: Main Effects Model
## Shown here for simple block design

$$Y = \mu + \alpha_k + \beta_j + \varepsilon$$

- $\mu$: Grand mean — Mean of all exams for all students
- $\alpha_k$: Effect for $k^{th}$ treatment — How mean for $k^{th}$ exam differs from overall mean
- $\beta_j$: Effect for $j^{th}$ block — How mean for $j^{th}$ student differs from overall mean
- $\varepsilon$: Random error

## Randomized Block—Calculations

1. Find the mean for each treatment (row means), each block (column means), and grand mean.
2. Partition the *SSTotal* into three pieces:

$$SSTotal = SSA + SSB + SSE$$

$SSTotal = \sum (y - \bar{y})^2 = (n-1)s_Y^2$ (As usual)

$SSA = \sum J(\bar{y}_k - \bar{y})^2$ Compare row means (exams)

$SSB = \sum K(\bar{y}_j - \bar{y})^2$ Compare column means (students)

$SSE = SSTotal - SSA - SSB$ (Unexplained error)

## Randomized Block ANOVA Table

| Source | d.f. | S.S. | M.S. | t.s. | p-value |
|---|---|---|---|---|---|
| Trts = A | *K*-1 | *SSTr* | *SSTr*/(*K*-1) | MSTr/MSE | |
| Blocks | *J*-1 | *SSB* | *SSB*/(*J*-1) | MSB/MSE | |
| Error | (*K*-1)(*J*-1) | *SSE* | *SSE*/(*K*-1)(*J*-1) | | |
| Total | *N*-1 | *SSTotal* | | | |

Testing TWO hypotheses:

$H_0$: $\alpha_1 = \alpha_2 = ... = \alpha_K = 0$
$H_a$: Some $\alpha_k \neq 0$ (Factor A: Difference in treatment means)

$H_0$: $\beta_1 = \beta_2 = ... = \beta_J = 0$
$H_a$: Some $\beta_j \neq 0$ (Factor B: Difference in block means)

## ANOVA Output in *R*

```
> BlockMod=aov(Grade~as.factor(Exam)+Student)
> summary(BlockMod)
                Df Sum Sq Mean Sq F value    Pr(>F)
as.factor(Exam)  3   1030  343.33  5.7222   0.01144 *
Student          4   4480 1120.00 18.6667 4.347e-05 ***
Residuals       12    720   60.00
```

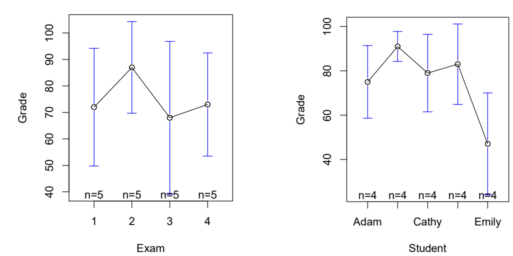
What if we ignored Blocks (Students) and treated it as a one-factor ANOVA? (See Lecture 15 – didn't take into account blocks!)

```
> model=aov(Grade~as.factor(Exam))
> summary(model)
               Df Sum Sq Mean Sq F value Pr(>F)
as.factor(Exam)  3 1030.0   343.3  1.0564  0.395
Residuals       16 5200.0   325.0
```

Ignoring "student effect," exams don't seem to differ; but including student effect, exams do differ. SS(Student) becomes part of SSE if Blocks are ignored, which *inflates* the estimate of the standard deviation.

## After Installing Three Packages in *R*: gplots, gdata, gtools

```
> plotmeans(Grade~Exam)
> plotmeans(Grade~Student)
```




95% CI's for each group mean are shown in blue.

## Fisher's LSD CIs After Two-Way ANOVA in a Simple Block Design

Same as one-way, but we know that

$$\frac{1}{n_i} + \frac{1}{n_j} = \begin{cases} 2/J & \text{for row means} \\ 2/K & \text{for column means} \end{cases}$$

For treatment (row) means: $LSD = t^* \sqrt{MSE}\sqrt{\frac{2}{J}}$

For block (column) means $LSD = t^* \sqrt{MSE}\sqrt{\frac{2}{K}}$

## Two-way ANOVA model with Interaction

$$Y = \mu + \alpha_k + \beta_j + \gamma_{kj} + \varepsilon$$

- Grand mean ($\mu$)
- Main effect for Factor A ($\alpha_k$)
- Main effect for Factor B ($\beta_j$)
- Interaction effect ($\gamma_{kj}$)
- Random error ($\varepsilon$)

## What's an Interaction Effect?

An *interaction effect* occurs when differences in mean level effects for one factor *depend* on the level of the other factor.

Example: $Y$ = GPA
Factor A = Year in School (FY, So, Jr, Sr)
Factor B = Major (Psych, Bio, Math)

FY is hard. $\Rightarrow \alpha_1 < 0$ (Main effect)

Bio is easy. $\Rightarrow \beta_2 > 0$ (Main effect)

Jr in Math is harder than just Jr or just Math $\Rightarrow \gamma_{33} < 0$ (Interaction effect)

## Example

Fire extinguishers tested to see how quickly they put out fires.

Factor A: 3 different chemicals in the extinguishers $A_1$, $A_2$, $A_3$
Factor B: 2 types of fires, $B_1$ = wood, $B_2$ = gas
$Y_{kj}$ = time to put out the fire of type $B_j$ with chemical $A_k$

Questions of interest:

- Do the 3 chemicals differ in mean time required? (If so, there is a *Factor A effect.*)
- Does mean time to put out fire depend on the type of fire? (If so, there is a *Factor B effect.*)
- Do the differences in times for the 3 chemicals depend on the type of fire? (If so, there is an *interaction* between chemical type and fire type.)

## Example: Putting out fires

Factor A Chemical ($A_1$, $A_2$, $A_3$)
Factor B Fire type (*wood, gas*)
Response: Time until fire is completely out (in seconds)

| Data: | Wood (j=1) | Gas (j=2) |
|---|---|---|
| A1 (k=1) | **52 64** | **72 60** |
| A2 (k=2) | **67 55** | **78 68** |
| A3 (k=3) | **86 72** | **43 51** |

$K = 3$
$J = 2$
$n = 2$
$N = 12$

## Interpreting Interaction

Cell means plot (Interaction plot)

| Data: | Wood | Gas |
|---|---|---|
| A1 | **58.0** | **66.0** |
| A2 | **61.0** | **73.0** |
| A3 | **79.0** | **47.0** |

## Interaction Plot via *R*

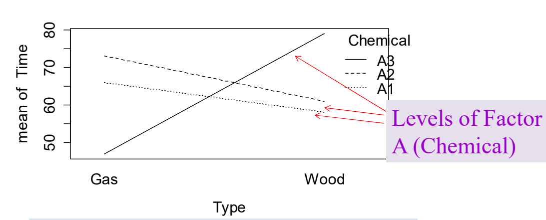
Generic

```
> interaction.plot(FactorA,FactorB,Y)
> interaction.plot(Chemical,Type,Time)
```

OR

```
> interaction.plot(Type,Chemical, Time)
```

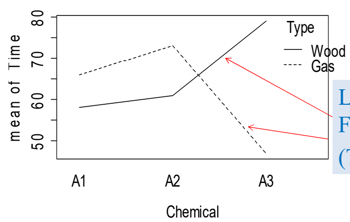
```
> interaction.plot(Type,Chemical,Time)
```




Levels of Factor B (Type of Fire)

Interpretation discussed in class.

```
> interaction.plot(Chemical,Type,Time)
```




Levels of Factor B (Type of Fire)

Levels of Factor A (Chemical)

Interpretation discussed in class.

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Two-way ANOVA (with Interaction)

$$Y = \mu + \alpha_k + \beta_j + \gamma_{kj} + \varepsilon$$

Grand mean

Main effect for Factor A

Main effect for Factor B

Interaction effect

Random error

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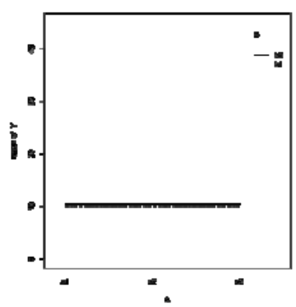
Generic Interaction Plots: 7 Cases

Suppose A has three levels and B has two levels.

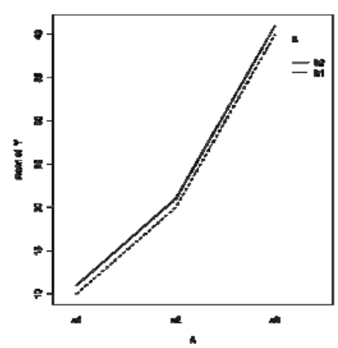
Grand mean only (no treatment effects)



$$Y = \mu + \varepsilon$$

---

Treatment A Effect, No Treatment B Effect

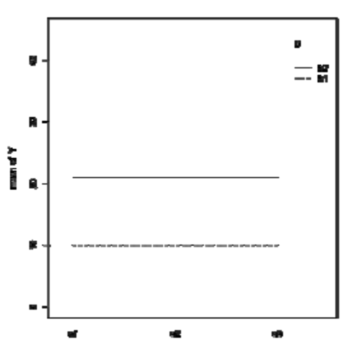


$$Y = \mu + \alpha_k + \varepsilon$$

Means differ for levels of A, *not* for levels of B

---

Treatment B Effect, No Treatment A Effect

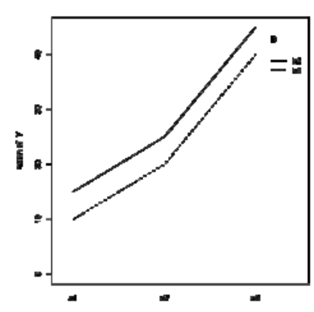


$$Y = \mu + \beta_j + \varepsilon$$

Means differ for levels of B, *not* for levels of A.

---

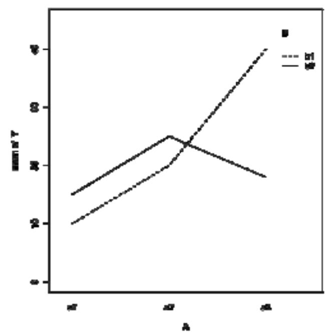
Treatments A and B Have Effects, No Interaction



$$Y = \mu + \alpha_k + \beta_j + \varepsilon$$

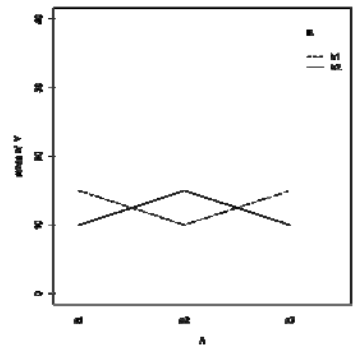
Differences between the 2 levels of Factor B don't depend on level of Factor A (and vice versa).

## A and B Effects Plus Interaction



$$Y = \mu + \alpha_k + \beta_j + \gamma_{kj} + \varepsilon$$

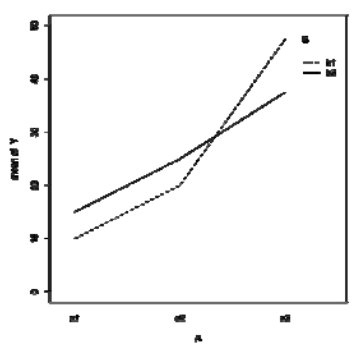
## No A Effect but a B Effect Plus Interaction



$$Y = \mu + \beta_j + \gamma_{kj} + \varepsilon$$

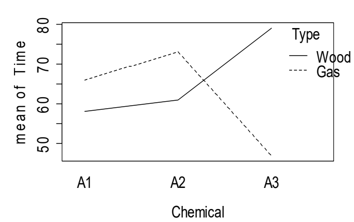
Cannot say means of A do not differ! They do, at *each* level of B.

## A Effect but No B Effect Plus Interaction



$$Y = \mu + \alpha_k + \gamma_{kj} + \varepsilon$$

Cannot say means of B do not differ! They do, at *each* level of A.




Looks like a *significant* interaction

*No* interaction: lines would be parallel.

Interaction: Differences in A *depend* on level of B.

However, looks like almost no *overall* A effect or B effect!

# Chapter 6 Section 6.3 The Gory Details!

## Recall: Main Effects Model

$$Y = \mu + \alpha_k + \beta_j + \varepsilon$$

- Grand Mean ($\mu$)
- Effect for $k^{th}$ level of Factor A ($\alpha_k$)
- Effect for $j^{th}$ level of Factor B ($\beta_j$)
- Random error ($\varepsilon$)

## Factorial Anova—Example: Putting out fires

Factor A: Chemical (A1, A2, A3)
Factor B: Fire type (*wood, gas*)
Response: Time required to put out fire (seconds)

| Data: | Wood | Gas | Row mean |
|---|---|---|---|
| A1 | **52 64** | **72 60** | **62** |
| A2 | **67 55** | **78 68** | **67** |
| A3 | **86 72** | **43 51** | **63** |
| Col mean | **66** | **62** | |

## Two-way ANOVA (with Interaction)

$$Y = \mu + \alpha_k + \beta_j + \gamma_{kj} + \varepsilon$$

- $\mu$: Grand mean
- $\alpha_k$: Main effect for Factor A
- $\beta_j$: Main effect for Factor B
- $\gamma_{kj}$: Interaction effect
- $\varepsilon$: Random error

## Factorial Design

Assume:

Factor A has $K$ levels, Factor B has $J$ levels.

To estimate an interaction effect, we need *more than one* observation for each combination of factors.

Let $n_{kj}$ = sample size in $(k,j)^{\text{th}}$ cell.

Definition: For a balanced design, $n_{kj}$ is constant for all cells.

$n_{kj} = n$

- $n = 1$ in a typical randomized block design
- $n > 1$ in a balanced factorial design

## Fire Extinguishers

Factor A: Chemical (*A1, A2, A3*)
Factor B: Fire type (*wood, gas*)
Response: Time required to put out fire (seconds)

| Data: | Wood | Gas |
|---|---|---|
| A1 | **52 64** | **72 60** |
| A2 | **67 55** | **78 68** |
| A3 | **86 72** | **43 51** |

$K = 3$
$J = 2$
$n = 2$
$N = 12$

## Estimating Factorial Effects

$\bar{y}_{kj}$ = mean for $(k,j)^{th}$ cell

$\bar{y}_k$ = mean for $k^{th}$ row

$\bar{y}_j$ = mean for $j^{th}$ column

$\bar{y}$ = Grand mean

$$y = \mu + \alpha_k + \beta_j + \gamma_{kj} + \varepsilon$$

$$(y-\bar{y}) = (\bar{y}_k - \bar{y}) + (\bar{y}_j - \bar{y}) + (\bar{y}_{kj} - \bar{y}_k - \bar{y}_j + \bar{y}) + (y - \bar{y}_{kj})$$

Total = Factor A + Factor B + Interaction + Error

$$SSTotal = SSA + SSB + SSAB + SSE$$

## Partitioning Variability (Balanced)

$SSTotal = \sum (y-\bar{y})^2 = (N-1)s_Y^2$ (As usual)

$SSA = \sum_k Jn(\bar{y}_k - \bar{y})^2$ (Row means)

$SSB = \sum_j Kn(\bar{y}_j - \bar{y})^2$ (Column means)

$SSAB = \sum_{k,j} n(\bar{y}_{kj} - \bar{y}_k - \bar{y}_j + \bar{y})^2$ (Cell means)

$SSE = \sum (y - \bar{y}_{kj})^2 = SSTotal - SSA - SSB - SSAB$

$SSTotal = SSA + SSB + SSAB + SSE$ (Error)

Total = Factor A + Factor B + Interaction + Error

## Decomposition: Fire extinguishers

| Data: | Wood | Gas |
|---|---|---|
| A1 | **52 64** | **72 60** |
| A2 | **67 55** | **78 68** |
| A3 | **86 72** | **43 51** |

| Cell Means: | Wood | Gas | Row mean | Trt A effect |
|---|---|---|---|---|
| A1 | **58.0** | **66.0** | **62** | **−2** |
| A2 | **61.0** | **73.0** | **67** | **+3** |
| A3 | **79.0** | **47.0** | **63** | **−1** |
| Col mean | **66** | **62** | ***64*** | |
| Trt B effect | **+2** | **−2** | | |

| Interaction effects | Wood | Gas | Row mean | Trt A effect |
|---|---|---|---|---|
| A1 | **−6** | **6** | **62** | **−2** |
| A2 | **−8** | **8** | **67** | **+3** |
| A3 | **14** | **−14** | **63** | **−1** |
| Col mean | **66** | **62** | ***64*** | |
| Trt B effect | **+2** | **−2** | | |

58 – 66 – 62 + 64 = −6

## Decomposition: Fire Extinguishers

Top left cell: **52,64** Top right cell: **72,60**

| Observed Value | | Grand Mean | | Trt A Effect | | Trt B Effect | | Inter-action | | Residual |
|---|---|---|---|---|---|---|---|---|---|---|
| **52** | = | 64 | + | -2 | + | **2** | + | -6 | + | **-6** |
| **64** | = | 64 | + | -2 | + | **2** | + | -6 | + | **6** |
| **72** | = | 64 | + | -2 | + | **-2** | + | 6 | + | **6** |
| **60** | = | 64 | + | -2 | + | **-2** | + | 6 | + | **-6** |

Etc.

## Two-way ANOVA Table (with Interaction)

| Source | d.f. | S.S. | M.S. | t.s. | p |
|---|---|---|---|---|---|
| Factor A | $K-1$ | $SSA$ | $SSA/(K-1)$ | **MSA/MSE** | |
| Factor B | $J-1$ | $SSB$ | $SSB/(J-1)$ | **MSB/MSE** | |
| A × B | $(K-1)(J-1)$ | $SSAB$ | $SSAB$/df | **MSAB/MSE** | |
| Error | $KJ(n-1)$ | $SSE$ | $SSE$/df | | |
| Total | $N-1$ | $SSY$ | | | |

$H_0$: $\alpha_1 = \alpha_2 = \ldots = \alpha_K = 0$

$H_0$: $\beta_1 = \beta_2 = \ldots = \beta_J = 0$

$H_0$: All $\gamma_{kj} = 0$

(Looking back) If $n = 1$ then df(interaction) = 0

Recall: Randomized Block ANOVA Table

| Source | d.f. | S.S. | M.S. | t.s. | p-value |
|---|---|---|---|---|---|
| **Trts/A** | K−1 | ***SSTr*** | ***SSTr*/(*K*−1)** | **MSTr/MSE** | |
| **Block** | **J−1** | ***SSB*** | ***SSB*/(*J*−1)** | **MSB/MSE** | |
| **Error** | (K−1)(J−1) | ***SSE*** | ***SSE*/(*K*−1)(*J*−1)** | | |
| **Total** | N−1 | ***SSTotal*** | | | |

## Fire Example: Two-way ANOVA Table, with Interaction

| Source | d.f. | S.S. | M.S. | t.s. | p |
|---|---|---|---|---|---|
| Chemical | 2 | 56 | 28.0 | 0.42 | 0.672 |
| Type | 1 | 48 | 48.0 | 0.73 | 0.426 |
| A × B | 2 | 1184 | 592.0 | 8.97 | 0.016 |
| Error | 6 | 396 | 66.0 | | |
| Total | 11 | 1684 | | | |

$H_0$: $\alpha_1 = \alpha_2 = \ldots = \alpha_K = 0$

$H_0$: $\beta_1 = \beta_2 = \ldots = \beta_J = 0$

$H_0$: All $\gamma_{kj} = 0$

## Two-way ANOVA with *R*

Option #1 - aov

```
> model=aov(Time~Chemical+Type+Chemical:Type)
> summary(model)
              Df Sum Sq Mean Sq F value Pr(>F)
Chemical       2     56      28   0.424 0.6725
Type           1     48      48   0.727 0.4265
Chemical:Type  2   1184     592   8.970 0.0157 *
Residuals      6    396      66
```

## Two-way ANOVA with *R*

Option #2 – anova(lm), when predictors are categorical

```
> anova(lm(Time~Chemical+Type+Chemical:Type))
Analysis of Variance Table

Response: Time
              Df Sum Sq Mean Sq F value   Pr(>F)
Chemical       2     56      28  0.4242 0.67247
Type           1     48      48  0.7273 0.42649
Chemical:Type  2   1184     592  8.9697 0.01574 *
Residuals      6    396      66
```

If sample sizes are <u>not</u> equal, order matters.

New example (on website):
Y = GPA
Explanatory variables are:
Seat location (front, middle, back)
Alcohol consumption
(none, some, lots)