Lecture 11

Review Section 3.5 from last Monday (on board)

Overview of today's example (on board)

Section 3.6, Continued: Nested F tests, review on board first

Section 3.4:

Interaction for quantitative variables (on board)

Polynomial Regression

Especially quadratic

Second-order models (including interaction)

Example: 1982 State SAT Scores (First year state by state data available)

Unit = A state in the United States

Response Variable:

Y = Average combined SAT Score

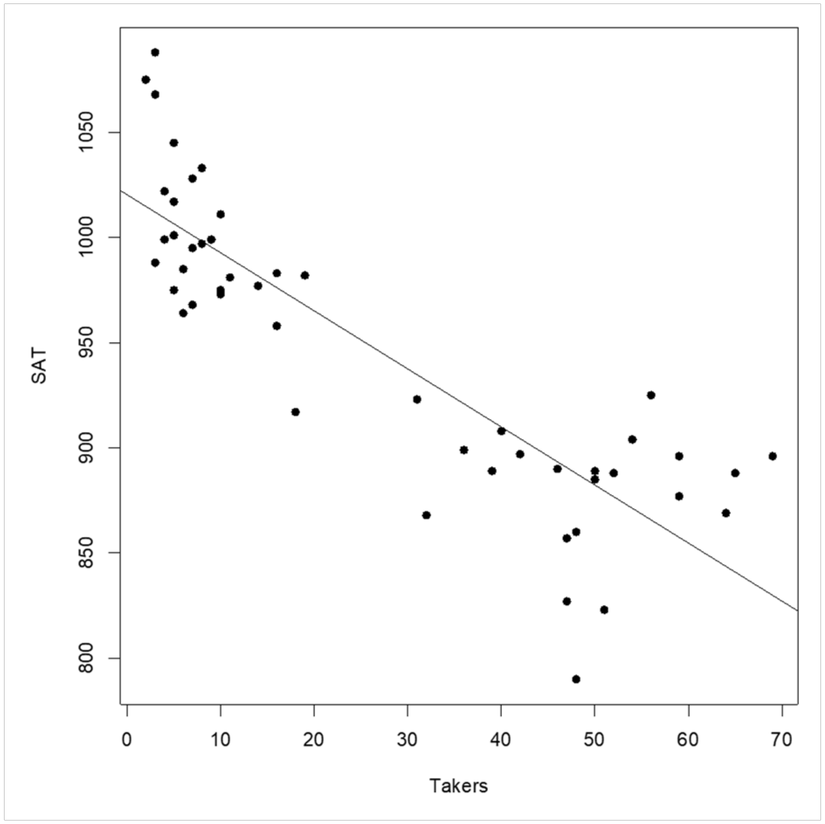
Potential Predictors:

X_1 = Takers = % taking the exam out of all eligible students in that state

X_2 = Expend = amount spent by the state for public secondary schools, per student ($100's)

Is *Y* related to one or both of these X variables?

Example: State SAT with X_1 only



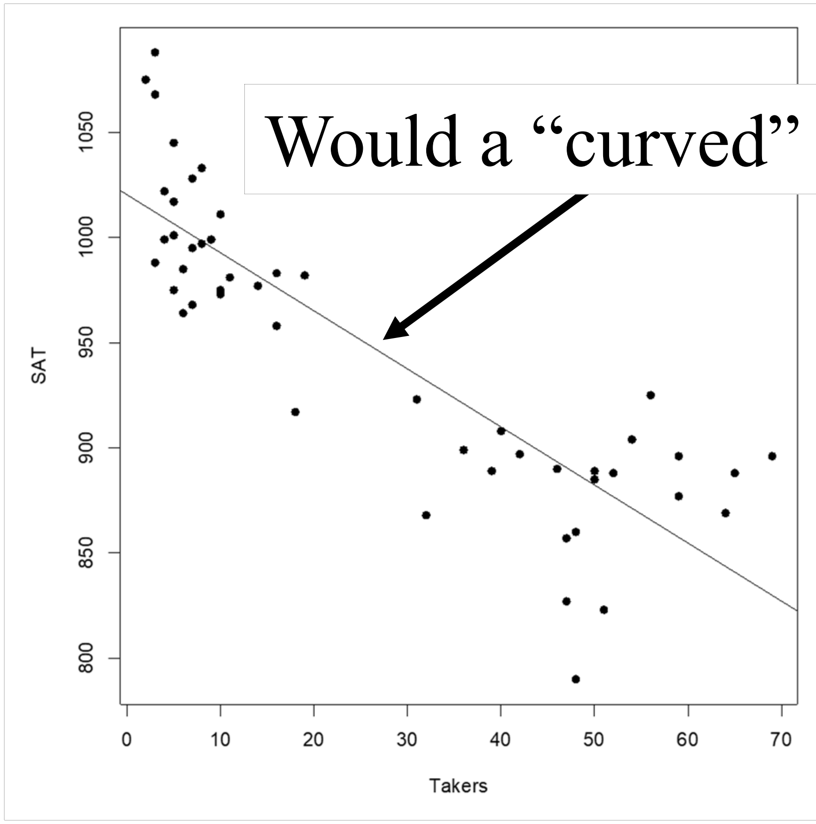
$Y =$ Combined SAT

$X =$ % Taking SAT

Things to notice:

- Two clusters in the X range. Why?
- Possible curved relationship
- As %Takers goes up, average SAT goes *down*.

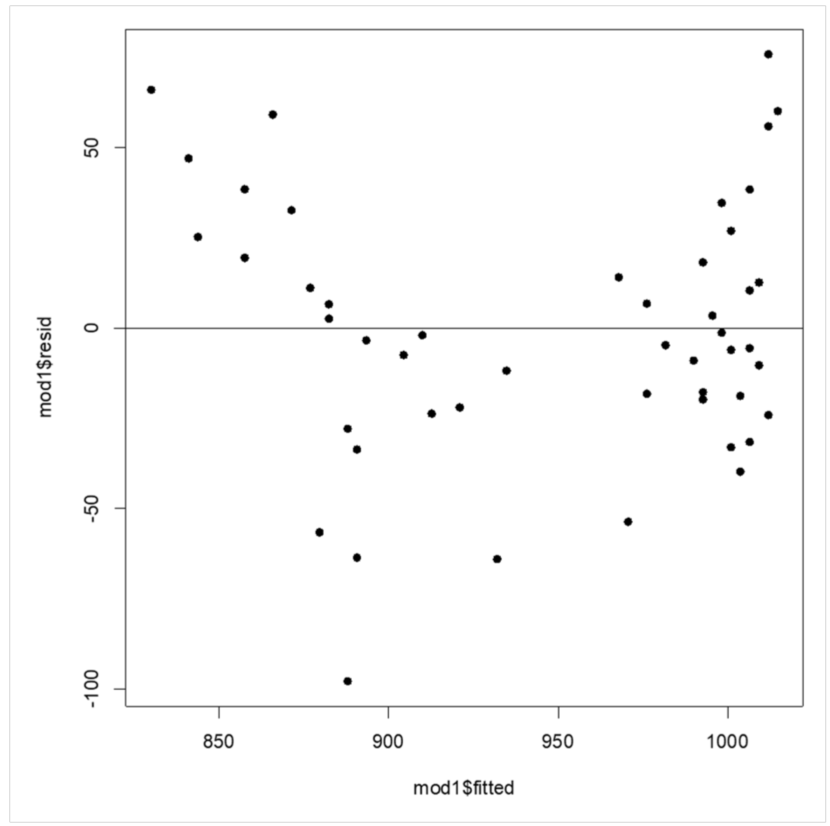
Example: State SAT with X_1 only



Would a "curved" line work better?

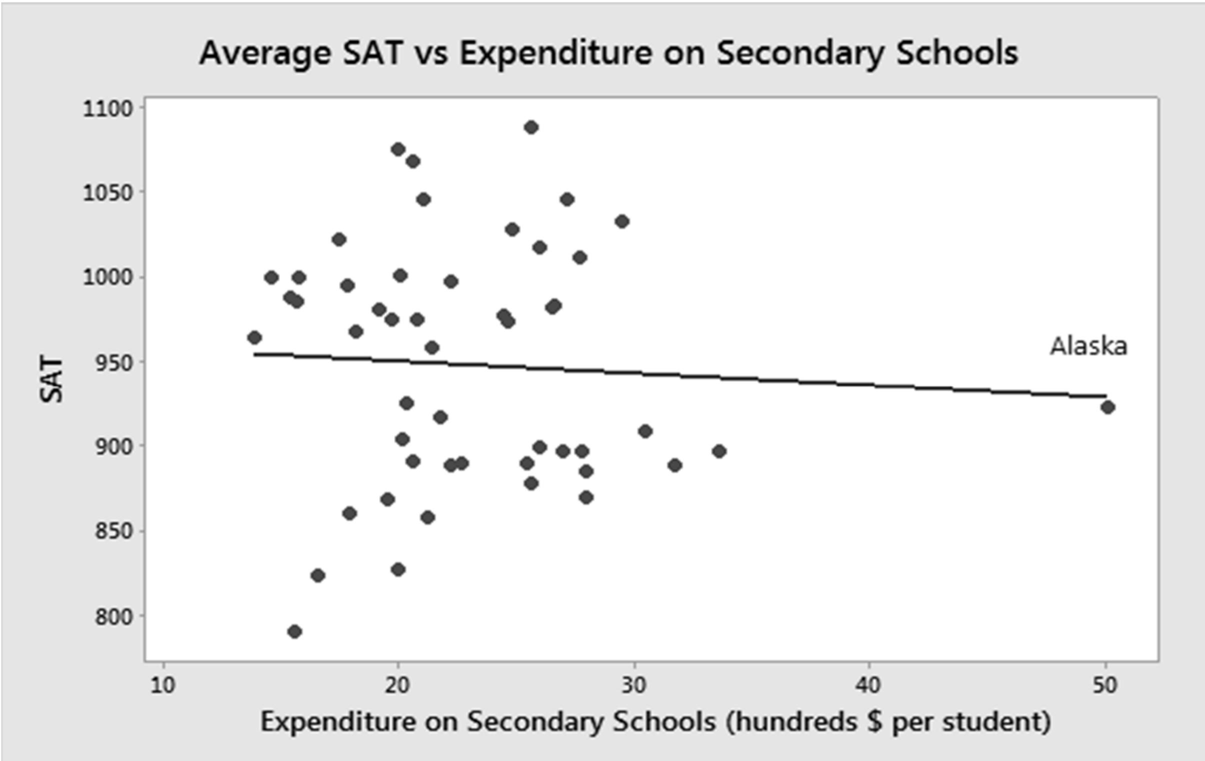
$Y =$ Combined SAT

$X =$ % Taking SAT



Residuals vs fitted values

Example: State SAT with X_2 only



Y = Combined SAT
X = Expenditure

Not clear what pattern is; Alaska is very influential.

Polynomial Regression

For a single predictor X:

$$Y = \beta_o + \beta_1 X + \beta_2 X^2 + \cdots + \beta_p X^p + \varepsilon$$

$$Y = \beta_o + \beta_1 X + \varepsilon$$ (Linear)

$$Y = \beta_o + \beta_1 X + \beta_2 X^2 + \varepsilon$$ (Quadratic; curve)

$$Y = \beta_o + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \varepsilon$$ (Cubic)

Polynomial Regression in *R*

Method #1: Create new columns with powers of the predictor.

To avoid creating a new column…

Method #2: Use **`I( )`** in the **`lm( )`**

```
quadmod=lm(SAT~Takers+I(Takers^2))
```

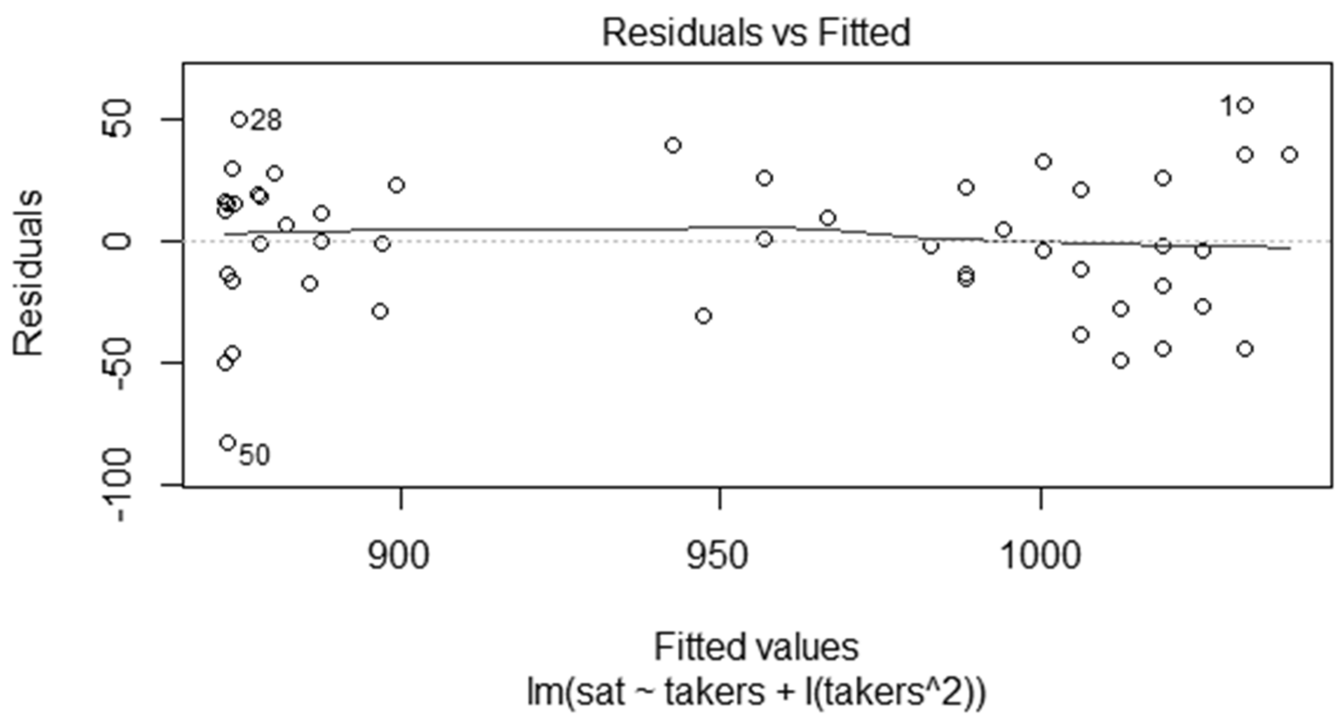
Method #3: Use **`poly`**

```
quadmod=lm(SAT~poly(Takers,degree=2,raw=TRUE))
```

Quadratic Model

```
> Quad<-lm(sat~takers+I(takers^2), data=StateSAT)
> summary(Quad)
Call:
lm(formula = sat ~ takers + I(takers^2), data = StateSAT)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) 1053.13112    9.27372 113.561  < 2e-16 ***
takers        -7.16159    0.89220  -8.027 2.32e-10 ***
I(takers^2)    0.07102    0.01405   5.055 6.99e-06 ***
Residual standard error: 29.93 on 47 degrees of freedom
Multiple R-squared:  0.8289,  Adjusted R-squared:  0.8216
F-statistic: 113.8 on 2 and 47 DF,  p-value: < 2.2e-16
```

Residual Plot Looks Good
(Two clusters still obvious)



How to Choose the Polynomial Degree?

- Use the minimum degree needed to capture the structure of the data.
- Check the t-test for the highest power.
- (Generally) keep lower powers—even if not "significant."

Interaction

Recall:

$$Active = \beta_o + \beta_1 Rest + \beta_2 Gender + \beta_3 Rest * Gender + \varepsilon$$

Product allows for different Active/Rest slopes for different genders

In General: Interaction is present if the relationship between two variables (e.g. Y and X_1) changes depending on a third variable (e.g. X_2).

Modeling tip: Include a product term to account for interaction.

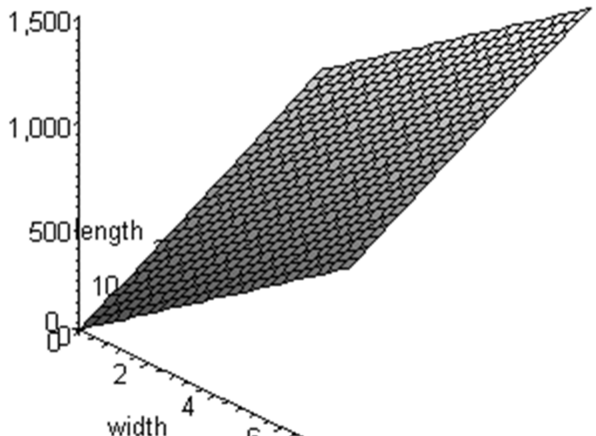
Complete Second-order Models

Definition: A complete second-order model for two predictors would be:

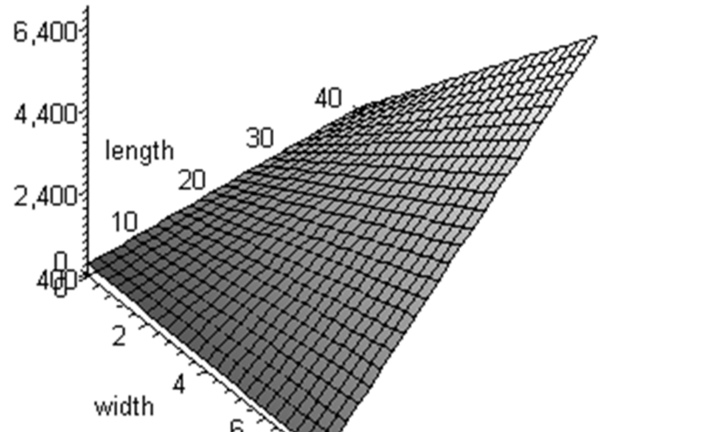
$$Y = \beta_o + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1^2 + \beta_4 X_2^2 + \beta_5 X_1 X_2 + \varepsilon$$

First order Quadratic Interaction

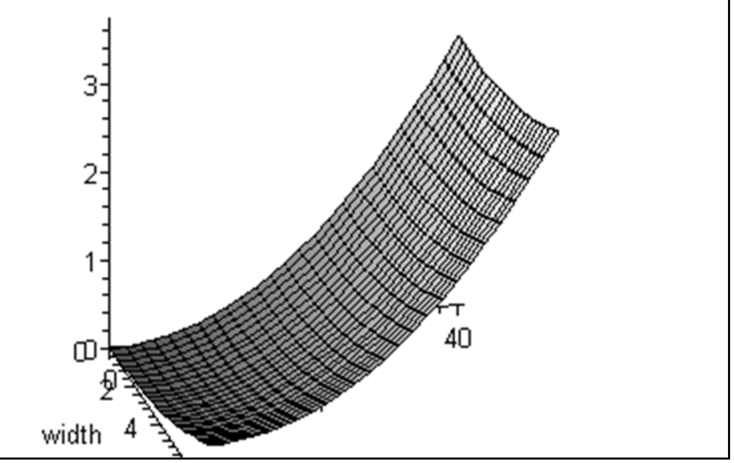
Just linear for both



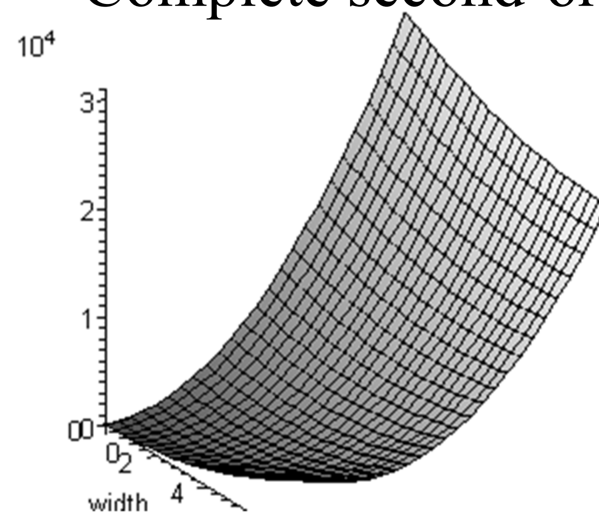
Linear + X_1 X_2 Interaction



Linear + Quadratic



Complete second-order



Second-order Model for State SAT

Example: Try a full second-order model for Y = SAT using X_1 = Takers and X_2 = Expend.

$$Y = \beta_o + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1^2 + \beta_4 X_2^2 + \beta_5 X_1 X_2 + \varepsilon$$

```
secondorder=lm(SAT~Takers+I(Takers^2)
+Expend+I(Expend^2)+Takers:Expend,
data=StateSAT)
```

Second-order Model for State SAT

summary(secondorder)

```
lm(formula = sat ~ takers + I(takers^2) + expend + I(expend^2) +
    takers:expend, data = StateSAT)
Coefficients:
               Estimate Std. Error t value Pr(>|t|)
(Intercept)   893.66283   36.14094  24.727  < 2e-16 ***
takers         -7.05561    0.83740  -8.426 9.96e-11 ***
I(takers^2)     0.07725    0.01328   5.816 6.28e-07 ***
expend         10.33333    2.49600   4.140 0.000155 ***
I(expend^2)    -0.11775    0.04426  -2.660 0.010851 *
takers:expend  -0.03344    0.03716  -0.900 0.373087

Residual standard error: 23.68 on 44 degrees of freedom
Multiple R-squared:  0.8997,  Adjusted R-squared:  0.8883
F-statistic: 78.96 on 5 and 44 DF,  p-value: < 2.2e-16
```

Do we really need the quadratic terms?

Nested F-test

anova(secondorder) [FULL MODEL]

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
takers	1	181024	181024	322.8794	< 2.2e-16	***
I(takers^2)	1	22886	22886	40.8198	9.035e-08	***
expend	1	11700	11700	20.8678	3.956e-05	***
I(expend^2)	1	5278	5278	9.4148	0.003677	**
takers:expend	1	454	454	0.8098	0.373087	
Residuals	44	24669	561			

firstorder=lm(SAT~Takers*Expend) [REDUCED = NO QUADRATIC]

anova(firstorder)

Response: SAT

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
Takers	1	181024	181024	152.6279	3.245e-16	***
Expend	1	8709	8709	7.3428	0.009429	**
Takers:Expend	1	1720	1720	1.4499	0.234710	
Residuals	46	54558	1186			

anova(firstorder,secondorder) [COMPARE]

Model 1: SAT ~ Takers * Expend
Model 2: SAT ~ Takers + I(Takers^2) + Expend + I(Expend^2) + Takers:Expend

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)	
1	46	54558					
2	44	24669	2	29889	26.656	2.608e-08	***

The quadratic terms are significant as a pair (as well as individually).

Do we really need the terms with Expend? Nested F-test

Simultaneously test all three terms involving "Expend" in the second order model with "Takers" to predict SAT scores.

```
anova(secondordermodel)
```

Response: SAT

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Takers	1	181024	181024	322.8794	< 2.2e-16 ***
I(Takers^2)	1	22886	22886	40.8198	9.035e-08 ***
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Takers	1	181024	181024	322.8794	< 2.2e-16 ***
I(Takers^2)	1	22886	22886	40.8198	9.035e-08 ***
Expend	1	11700			
I(Expend^2)	1	5278			
Takers:Expend	1	454			
Residuals	44	24669			

17432 explained by adding the three predictors

$$t.s. = \frac{17432/3}{24669/44} = 10.36$$

Compare to F(3,44)

P-value=0.000028 (next slide)

anova(secondorder) [FULL MODEL]

	Df	Sum Sq	Mean Sq	F value	Pr(>F)	
takers	1	181024	181024	322.8794	< 2.2e-16	***
I(takers^2)	1	22886	22886	40.8198	9.035e-08	***
expend	1	11700	11700	20.8678	3.956e-05	***
I(expend^2)	1	5278	5278	9.4148	0.003677	**
takers:expend	1	454	454	0.8098	0.373087	
Residuals	44	24669	561			

Takersmodel=lm(SAT~Takers+I(Takers^2)) [REDUCED MODEL]

anova(Takersmodel)

Response: SAT

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Takers	1	181024	181024	202.089	< 2.2e-16 ***
I(Takers^2)	1	22886	22886	25.549	6.992e-06 ***
Residuals	47	42101	896		

anova(Takersmodel,secondorder) [COMPARE]

Model 1: SAT ~ Takers + I(Takers^2)
Model 2: SAT ~ Takers + I(Takers^2) + Expend + I(Expend^2) + Takers:Expend

	Res.Df	RSS	Df	Sum of Sq	F	Pr(>F)
1	47	42101				
2	44	24669	3	17432	10.364	2.787e-05 ***

Three “new” predictors reduce the SSE by 17432, a sig. amount.

Second-order Model for State SAT

summary(secondorder)

```
lm(formula = sat ~ takers + I(takers^2) + expend + I(expend^2) +

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Coefficients:
               Estimate Std. Error t value Pr(>|t|)
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Do we really need the interaction? T-test for takers:expend

SUMMARY

- Full second-order model is better than the model with no quadratic terms
- Full second-order model is better than the quadratic model with "Takers" only
- Model with no interaction seems acceptable
- Comparing Adjusted R-squared:
  - Full model: 88.83%; No interaction: 88.88%
  - No quadratic terms: 76.38%
  - Takers only, quadratic: 82.16%
  - Expend only, quadratic: -3.59%, partly because of the extreme outlier for Alaska!