Assumptions in the Normal Linear Regression Model

A1: There is a ***linear*** relationship between X and Y.

A2: The error terms (and thus the Y's at each X) have ***constant variance***.

A3: The error terms are ***independent***.

A4: The error terms (and thus the Y's at each X) are ***normally distributed*****.**
Note: In practice, we are looking for a fairly symmetric distribution with no major outliers.

Other things to check (**Q**uestions to ask):

Q5: Are there any major ***outliers*** in the data (X, or combination of (X,Y))?

Q6: Are there ***other possible predictors*** that should be included in the model?

Applet for illustrating the effect of outliers on the regression line and correlation: http://illuminations.nctm.org/LessonDetail.aspx?ID=L455

Useful Plots for Checking Assumptions and Answering These Questions

Reminders:

Residual = e_i = $Y_i - \hat{Y}_i$ = observed Y_i – predicted Y_i

Predicted Y_i = $\hat{Y}_i$ = $b_0 + b_1 X_i$, also called "fitted Y_i"

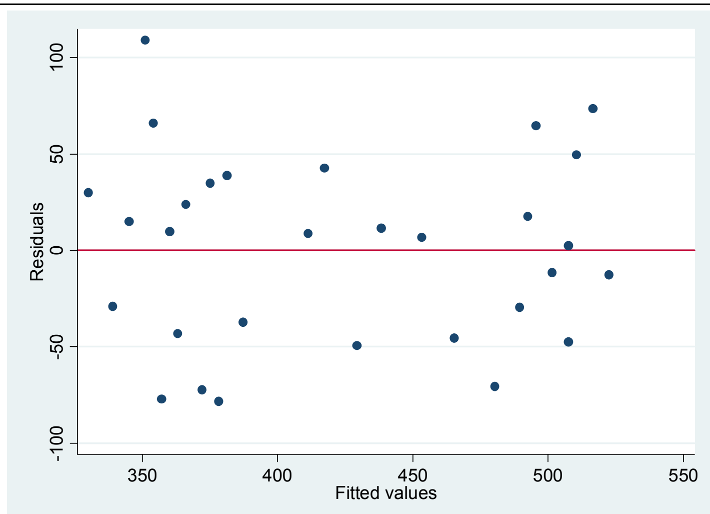
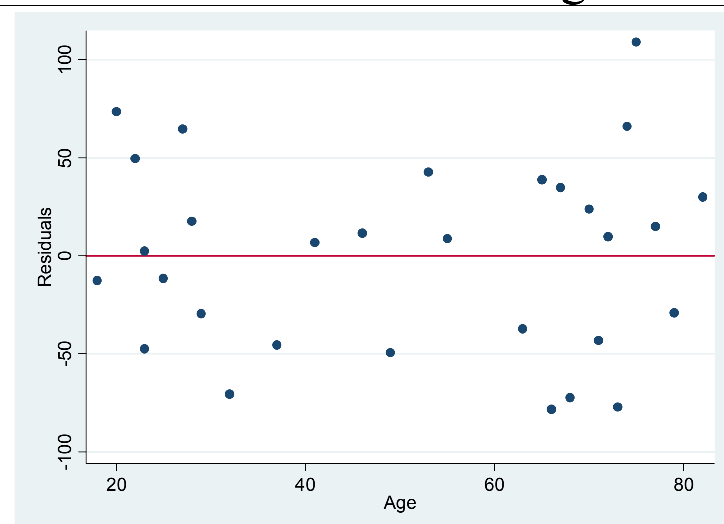
Definition: The *semi-studentized residual* for unit *i* is $e_i^* = \frac{e_i}{\sqrt{MSE}}$

Plot	Useful for
Dotplot, stemplot, histogram of X's	**Q5** Outliers in X; range of X values
Residuals e_i versus X_i or predicted $\hat{Y}_i$	**A1** Linear, **A2** Constant var., **Q5** outliers
SS resids e^* versus X_i or predicted $\hat{Y}_i$	As above, but a better check for outliers
Dotplot, stemplot, histogram of e_i	**A4** Normality assumption
Residuals e_i versus time (if measured)	**A3** Dependence across time
Residuals e_i versus other predictors	**Q6** Predictors missing from model
"Normal probability plot" of residuals	**A4** Normality assumption

Example: Highway sign data

Graph of residuals versus predicted ("fitted") values and residuals vs Age
Stata (following regress command): *rvfplot, yline(0)* and *rvpplot Age, yline(0)*

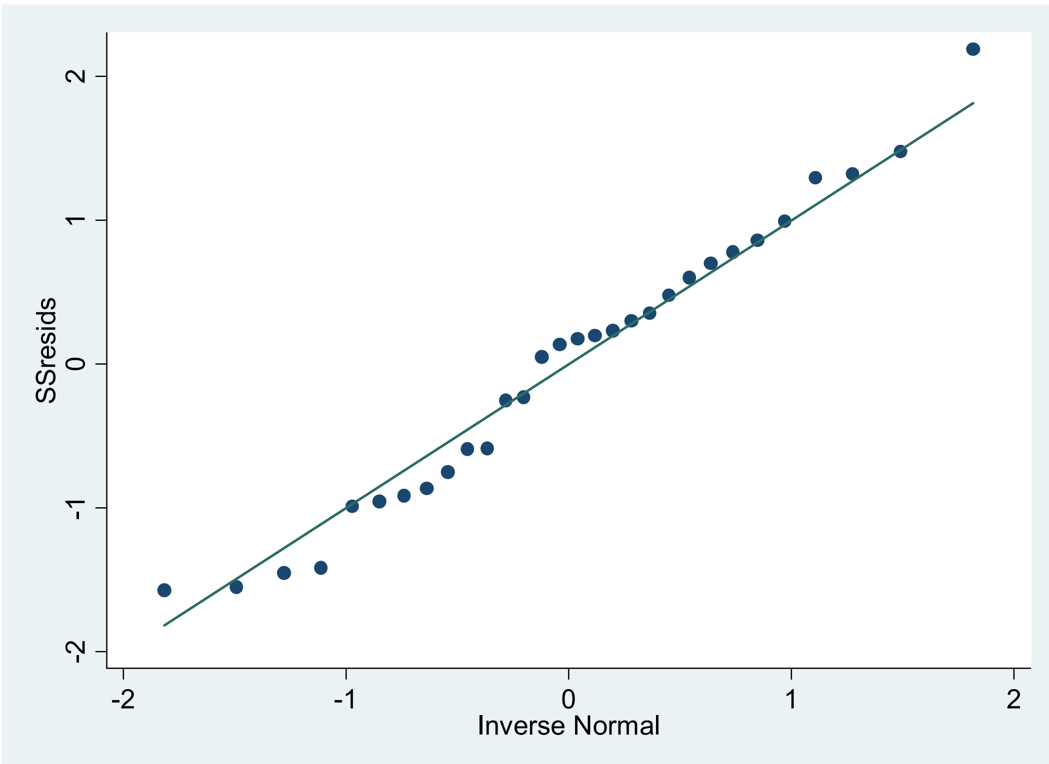
NOTE: Plot of residuals versus predictor variable X should look the same except for the scale on the X axis, because fitted values are linear transform of X's. However, when the slope is negative, one will be a mirror image of the other.

Residuals vs fitted values	Residuals vs age
	

Comments: These are good "residual plots." Points look randomly scattered around 0. No evidence of *nonlinear* pattern or *unequal variances*.

Some other plots of the residuals:

Normal probability plot of semi-studentized residuals (to check normality assumption, **A4**):



This is a pretty good plot. There is one point at each end that is slightly off, that might be investigated, but no major problems.
Stata command (following regress): *qnorm name* where "name" is what you named the semi-studentized residuals.

Stemplot of semi-studentized residuals (to check normality assumption):

stem SSresids

```
Stem-and-leaf plot for SSresids

SSresids rounded to nearest multiple of .01
plot in units of .01

 -1** | 57,55
 -1** | 45,42
 -0** | 99,96,91,87,75,59,59
 -0** | 25,23
  0** | 05,13,17,20,23,30,35,48
  0** | 60,70,78,86,99
  1** | 30,32,48
  1** |
  2** | 19
```

This is further confirmation that the residuals are relatively symmetric with no major outliers. The 2.19 is for a driver with X = 75 years, Y = 460 feet.

What to do when assumptions aren't met

Assumption 1:
Relationship is linear.

How to detect a problem:
Plot residuals versus fitted values. If you see a pattern, there is a problem with the assumption.

What to do about the problem:
Transform the **X** values, $X' = f(X)$. Then do the regression using X' instead of X:

$$Y = \beta_0 + \beta_1 X' + \varepsilon$$

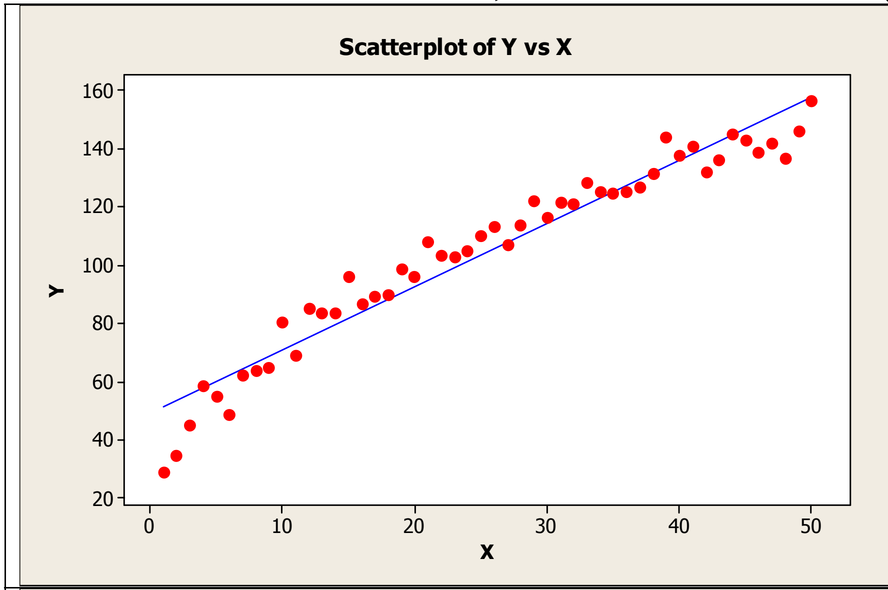
where we still assume the ε are $N(0, \sigma^2)$.

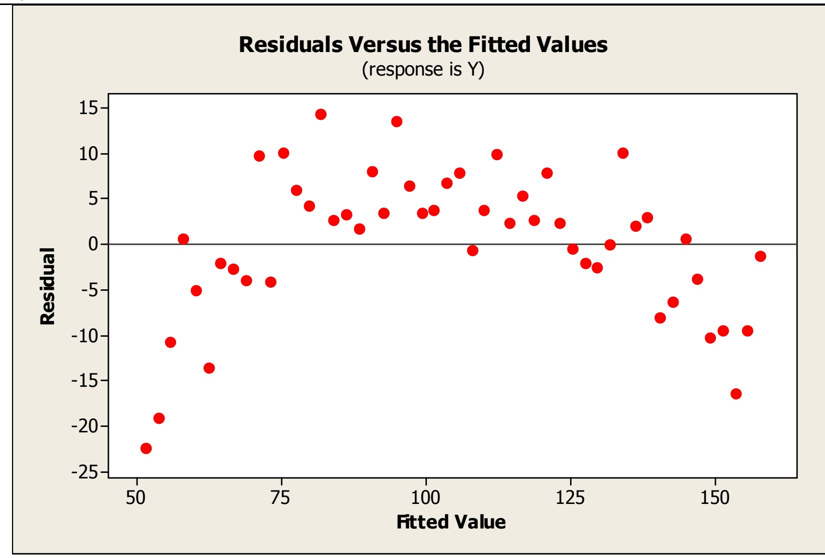
NOTE: Only use this "solution" if non-linearity is the *only* problem, not if it also looks like there is non-constant variance or non-normal errors. For those, we will transform Y.

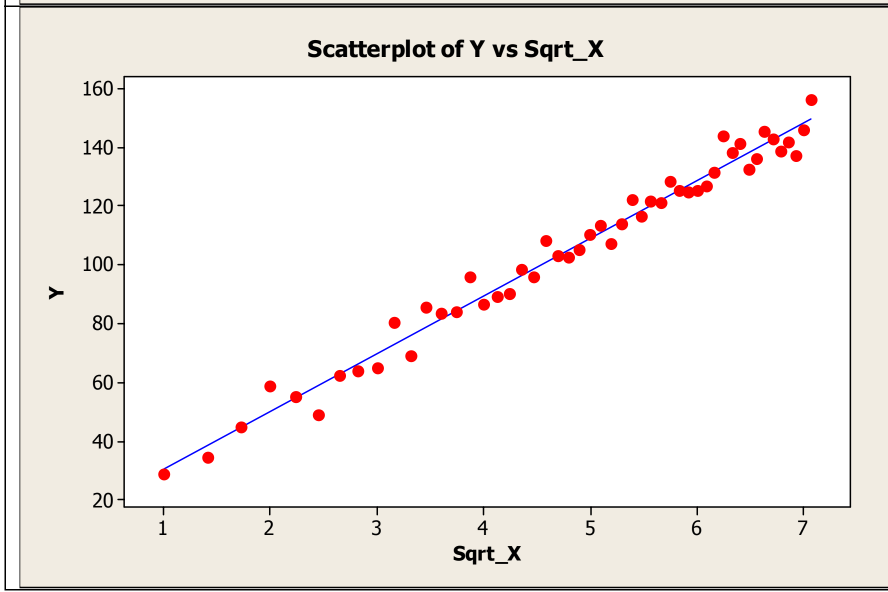
REASON: The errors are in the vertical direction. Stretching or shrinking the X-axis doesn't change those, so if they are normal with constant variance, they will stay that way.

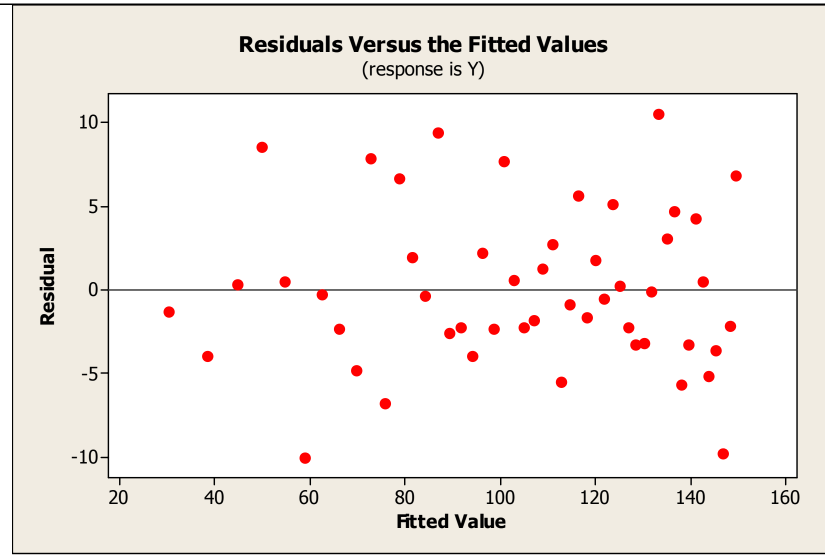
Let's look at what kinds of transformations to use. (Also see page 130 in textbook.)

Residuals are inverted U, use $X' = \sqrt{X}$ or $\log_{10} X$

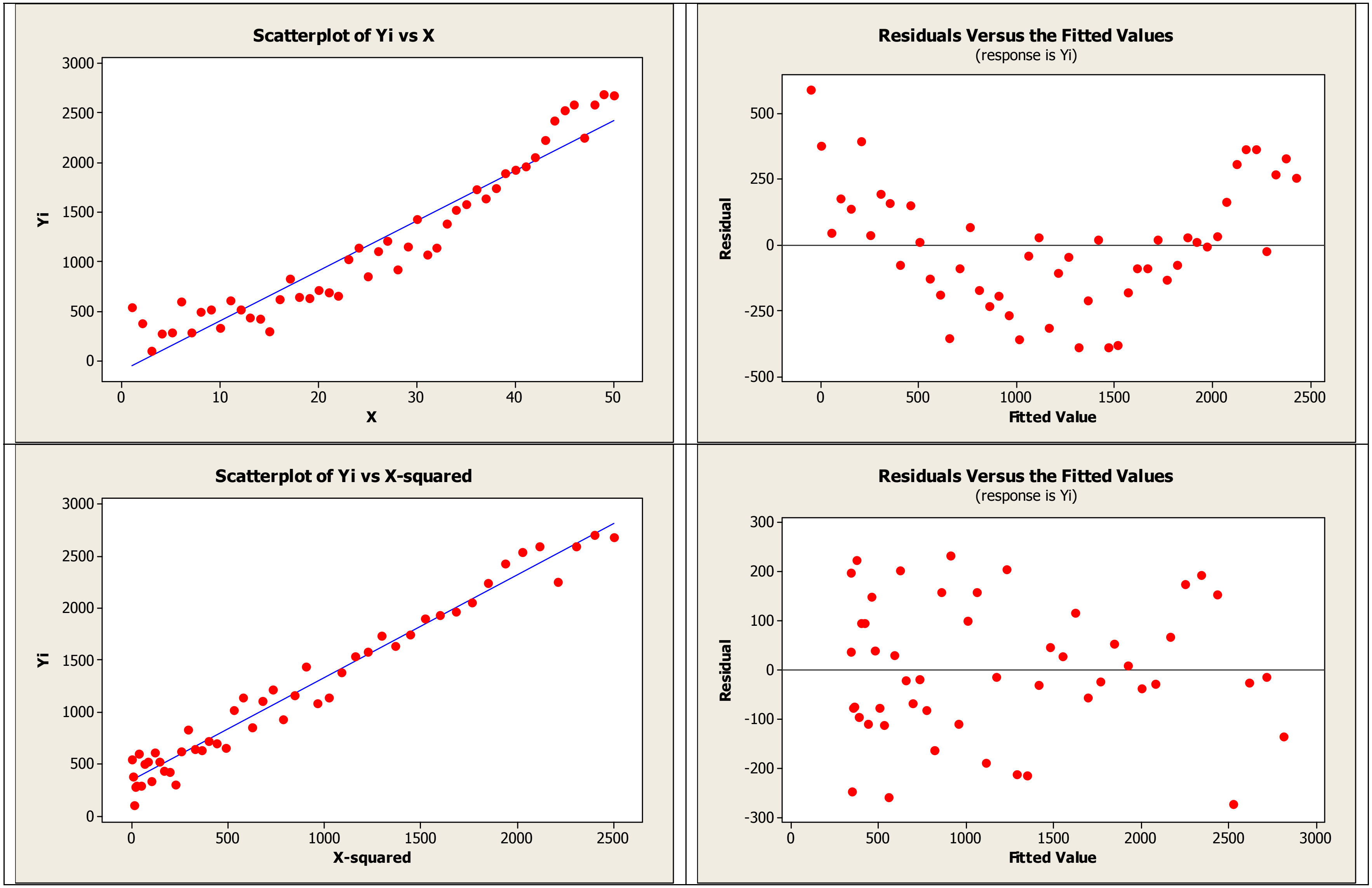




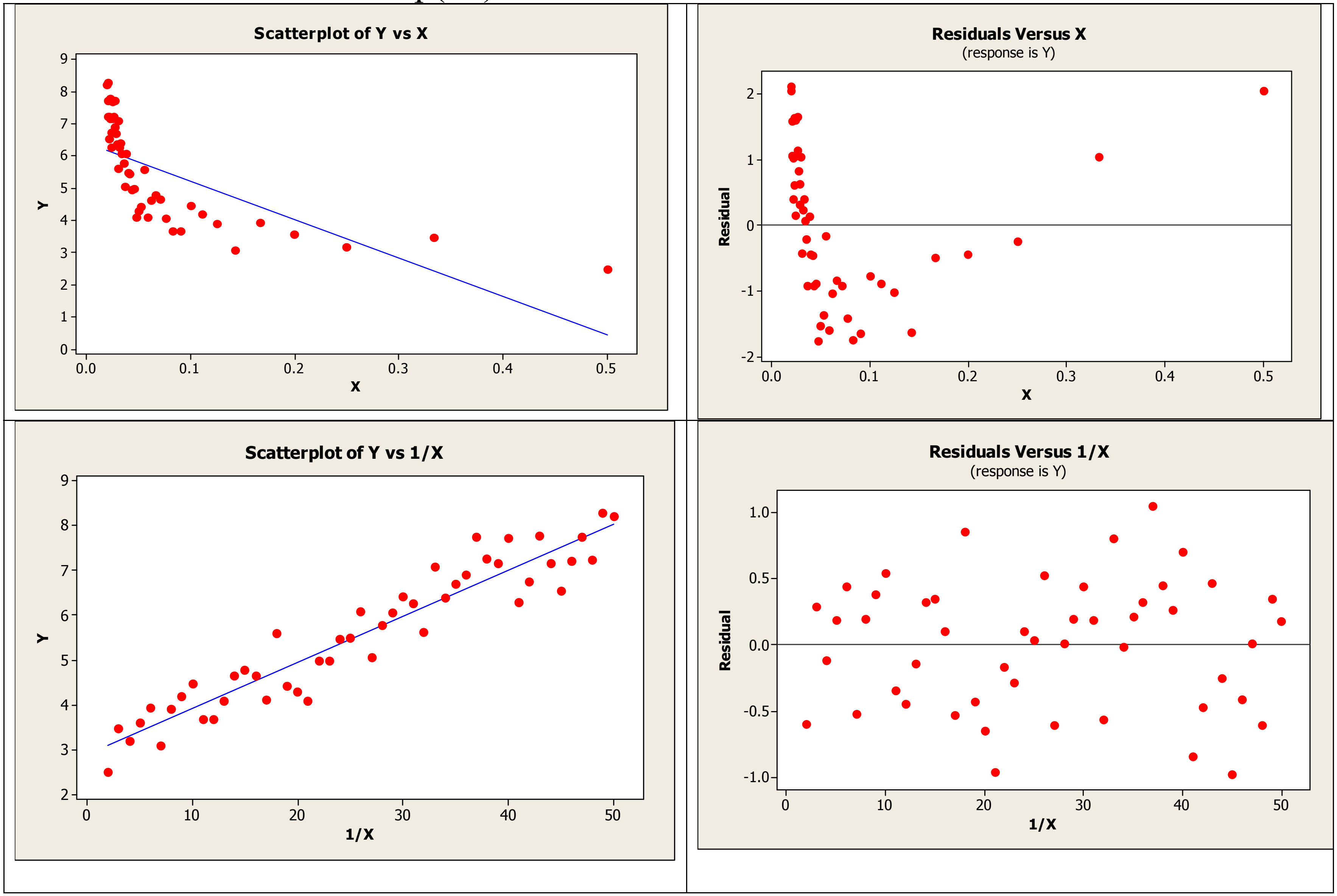




Residuals are U-shaped and association between X and Y is positive: Use X' = X^2



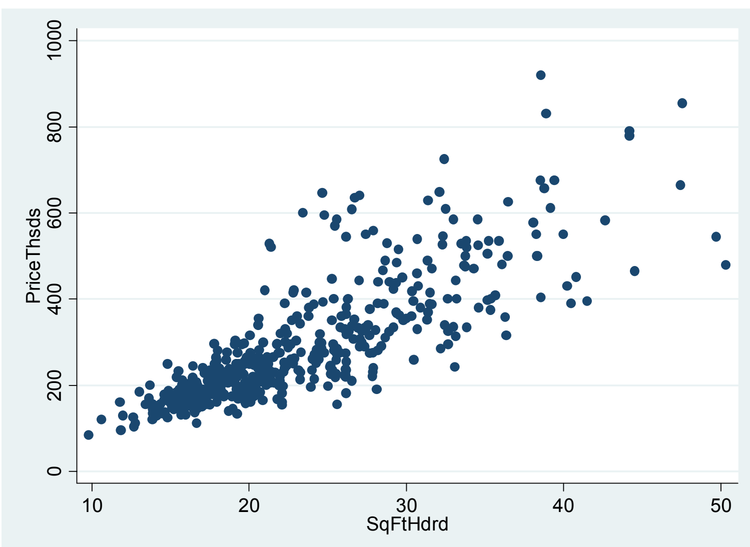
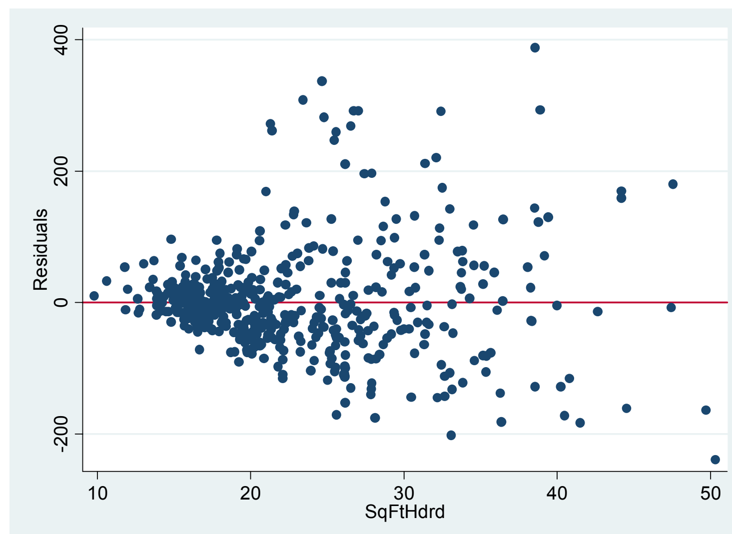
Residuals are U-shaped and association between X and Y is negative:
Use X' = 1/X   or   X' = exp(-X)



Assumption 2: Constant variance of the errors across X values.

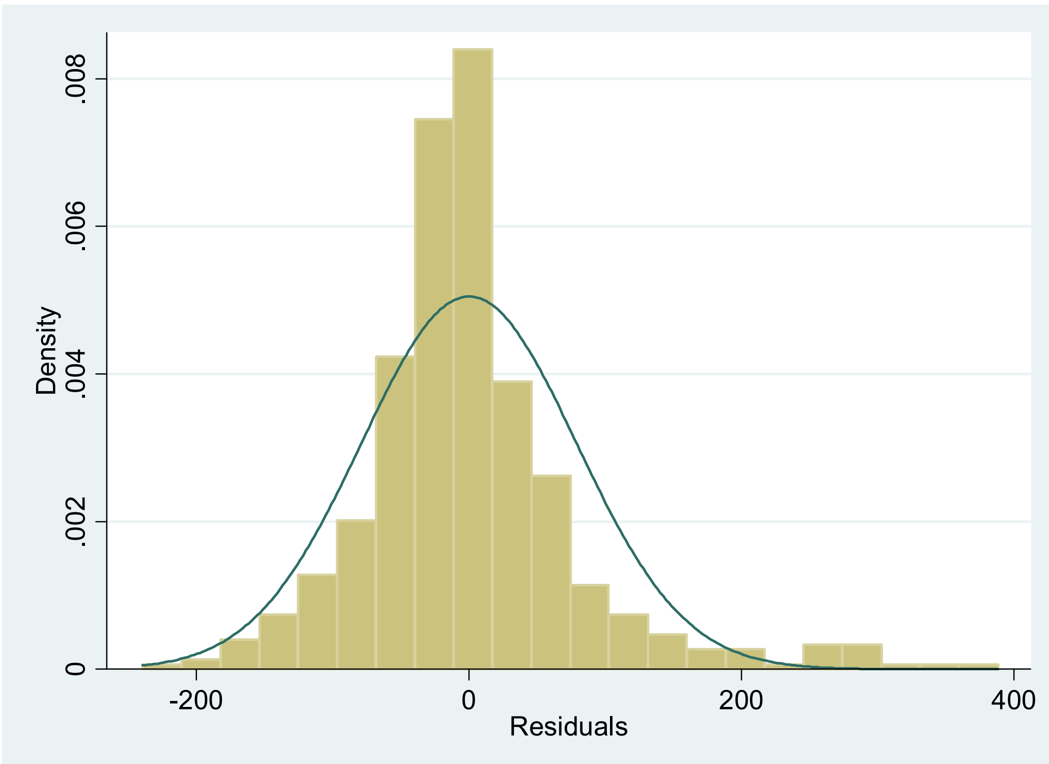
How to detect a problem:
Plot residuals versus fitted values. If you see increasing or decreasing spread, there is a problem with the assumption.

Example: Real estate data set C7 in Appendix C for n = 522 homes sold in a Midwestern city. Y = Sales price (thousands); X = Square feet (in hundreds).

Original data:	Residual plot:
	

Clearly, the variance is increasing as house size increases.

NOTE: Usually increasing variance and skewed distribution go together. Here is a histogram of the residuals, with a superimposed normal distribution. Notice the residuals extending to the right.

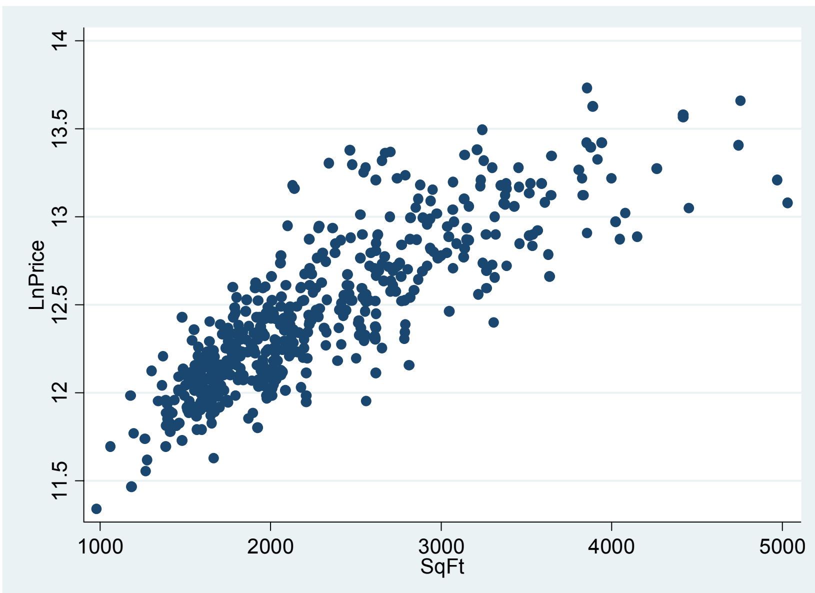


What to do about the problem:

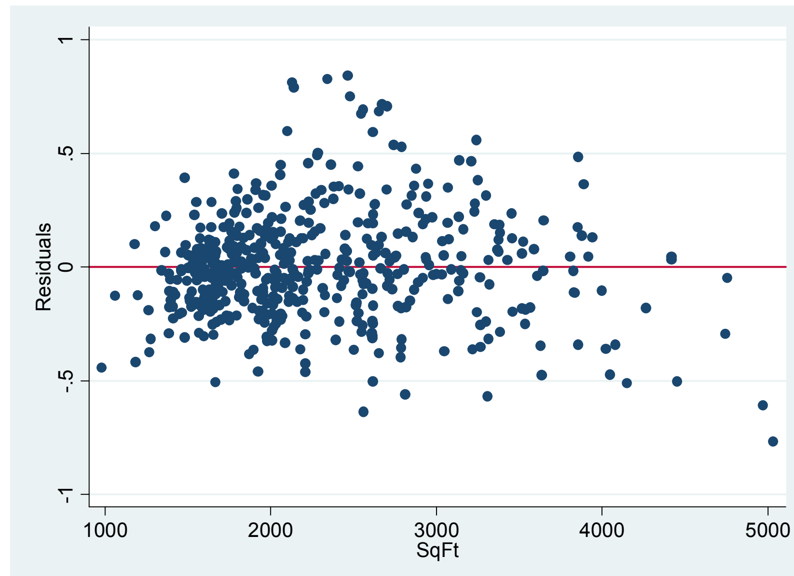
Transform the Y values, or both the X and Y values. See page 132 for pictures.

Example: Real estate sales, transform Y values to Y' = ln (Y)

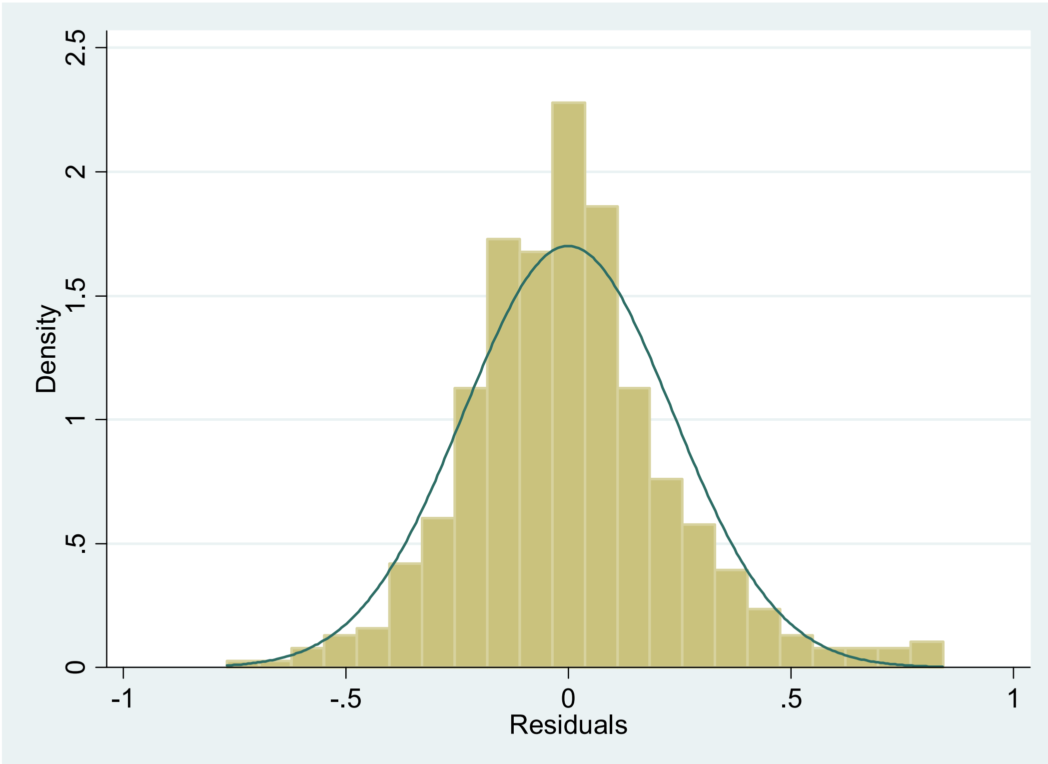
Scatter plot of ln(Price) vs Square feet



Residuals versus Square feet:



Looks like one more transformation might help – use square root of size. But we will leave it as this for now. See histogram of residual on next page.



This looks better – more symmetric and no outliers.

Using models after transformations

Transforming X only:

Use transformed X for future predictions: X' = f(X).
Then do the regression using X' instead of X:

$$Y = \beta_0 + \beta_1 X' + \varepsilon$$

where we still assume the ε are $N(0, \sigma^2)$.

For example, if $X' = \sqrt{X}$ then the predicted values are:

$$\hat{Y} = b_0 + b_1 \sqrt{X}$$

Transforming Y (and possibly X):

Everything must be done in transformed values. For confidence intervals and prediction intervals, get the intervals *first* and then transform the endpoints back to original units.

Example: Predicting house sales price using square feet. Regression equation is:

Predicted Ln(Price) = 11.2824 + 0.051(Square feet in hundreds)

For a house with 2000 square feet = 20 hundred square feet:

$$\hat{Y}' = 11.2824 + 0.051(20) = 12.3024$$

So predicted price = exp(12.3024) = $220,224.

95% prediction interval for Ln(Price) is 11.8402, 12.7634. Transform back to dollars:

Exp(11.8402) = $138,718
Exp(12.7634) = $349,200

95% confidence interval for the *mean* Ln(Price) is 12.2803, 12.3233

Exp(12.2803) = $215,410
Exp(12.3233) = $224,875

Assumption 3: Independent errors

1. The main way to check this is to understand how the data were collected. For example, suppose we wanted to predict blood pressure from amount of fat consumed in the diet. If we were to sample entire families, and treat them as independent, that would be wrong. If one member of the family has high blood pressure, related members are likely to have it as well. Taking a random sample is one way to make sure the observations are independent.

2. If the values were collected over time (or space) it makes sense to plot the residuals versus order collected, and see if there is a trend or cycle. See page 109 for examples.

OUTLIERS

Some reasons for outliers:

1. A mistake was made. If it's obvious that a mistake was made in recording the data, or that the person obviously lied, etc., it's okay to throw out an outlier and do the analysis without it. For example, a height of 7 inches is an obvious mistake. If you can't go back and figure out what it should have been (70 inches? 72 inches? 67 inches?) you have no choice but to discard that case.

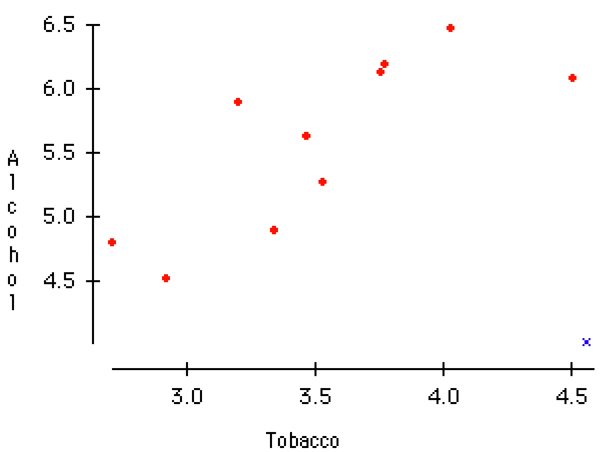
2. The person (or unit) belongs to a different population, and should not be part of the analysis, so it's okay to remove the point(s). An example is for predicting house prices, if a data set has a few mansions (5000+ square feet) but the other houses are all smaller (1000 to 2500 square feet, say), then it makes sense to predict sales prices for the smaller houses only. In the future when the equation is used, it should be used only for the range of data from which it was generated.

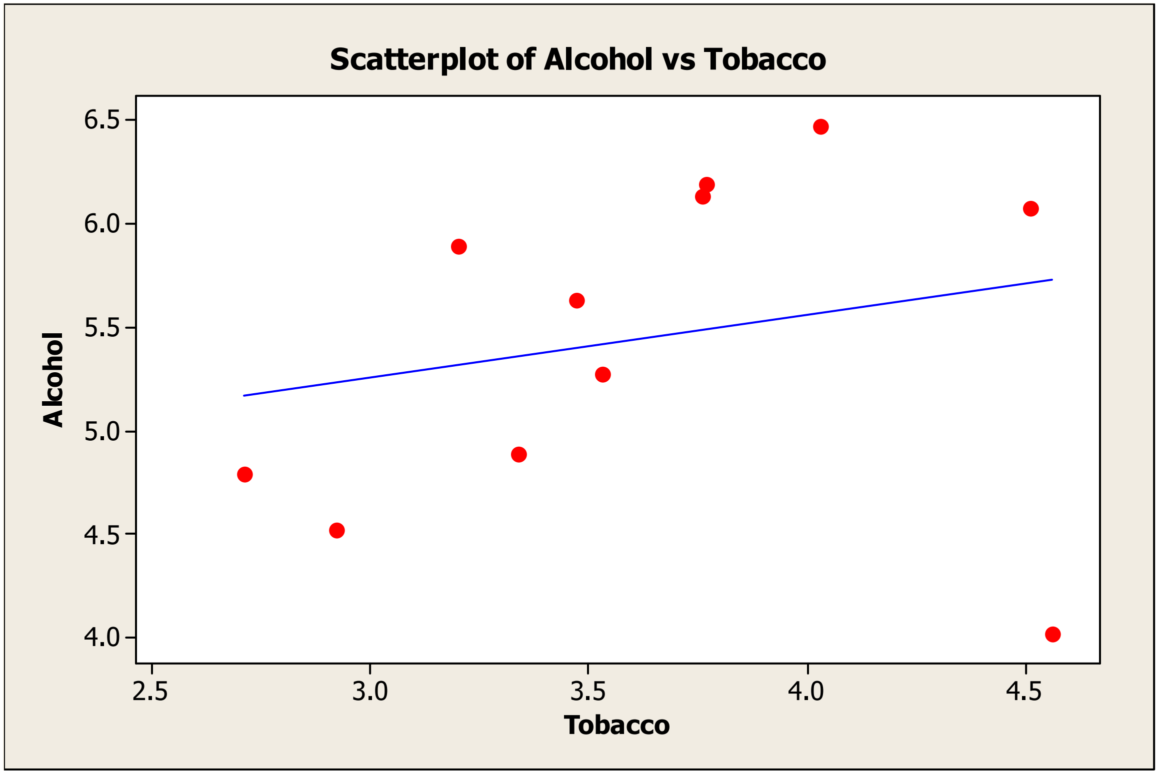
3. Sometimes outliers are simply the result of natural variability. In that case, it is NOT okay to discard them. If you do, you will underestimate the variance.

Story Name: Alcohol and Tobacco
Story Topics: Consumer , Health
Datafile Name: Alcohol and Tobacco
Methods: Correlation , Dummy variable , Outlier , Regression , Scatterplot
Abstract: Data from a British government survey of household spending may be used to examine the relationship between household spending on tobacco products and alcholic beverages. A scatterplot of spending on alcohol vs. spending on tobacco in the 11 regions of Great Britain shows an overall positive linear relationship with Northern Ireland as an outlier. Northern Ireland's influence is illustrated by the fact that the correlation between alcohol and tobacco spending jumps from .224 to .784 when Northern Ireland is eliminated from the dataset.

This dataset may be used to illustrate the effect of a single influential observation on regression results. In a simple regression of alcohol spending on tobacco spending, tobacco spending does not appear to be a significant predictor of tobacco spending. However, including a dummy variable that takes the value 1 for Northern Ireland and 0 for all other regions results in significant coefficients for both tobacco spending and the dummy variable, and a high R-squared.

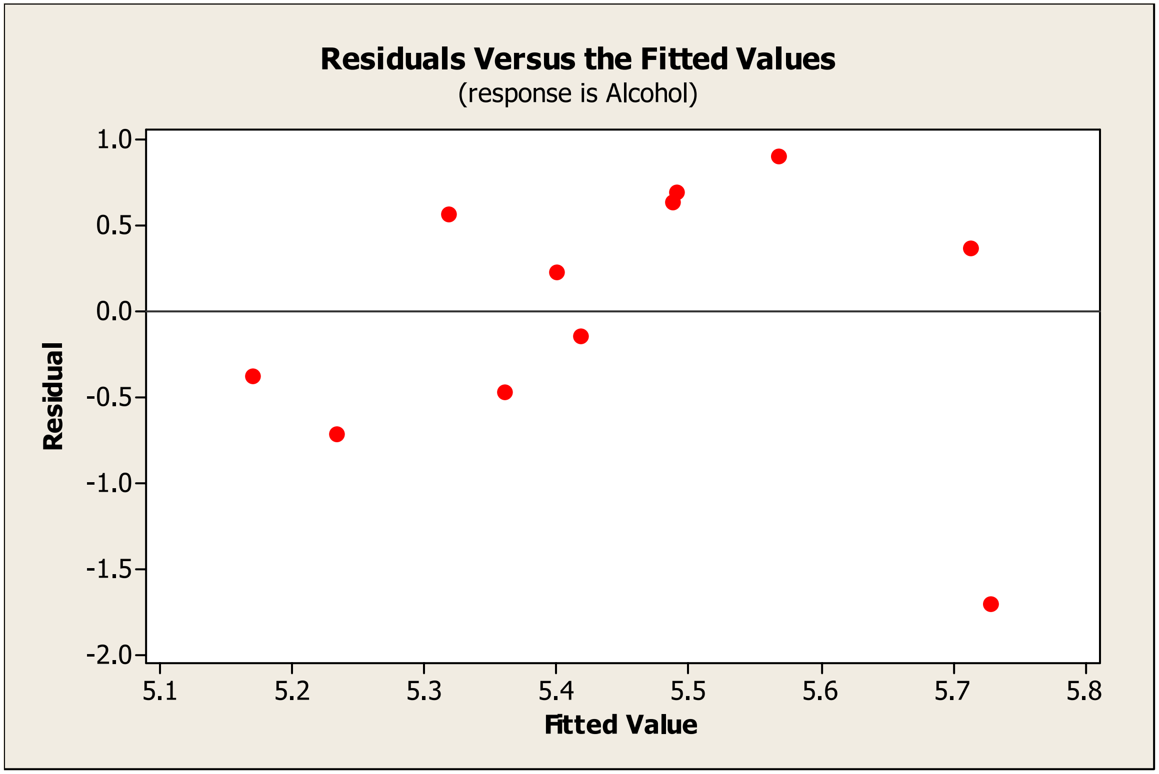
Image: Scatterplot of Alcohol vs. Tobacco, with Northern Ireland marked with a blue X.



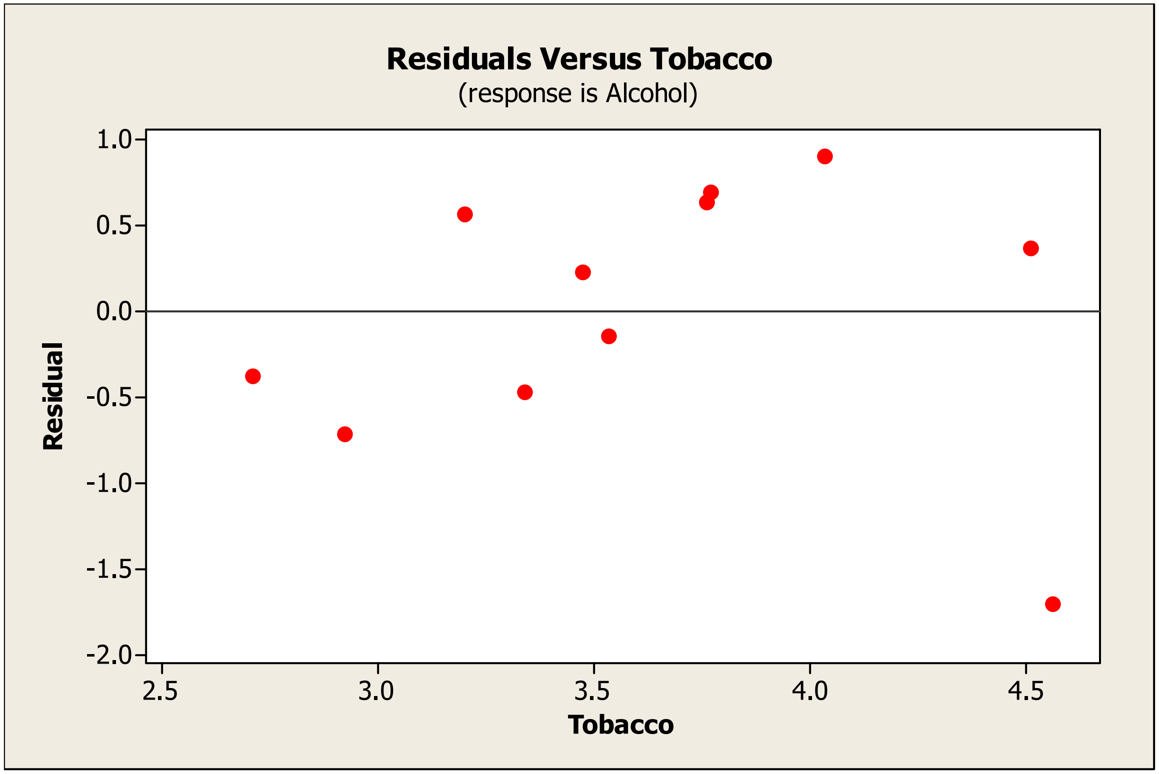


Notice Northern Ireland in lower right corner – a definite outlier, based on the combined (X,Y) values.

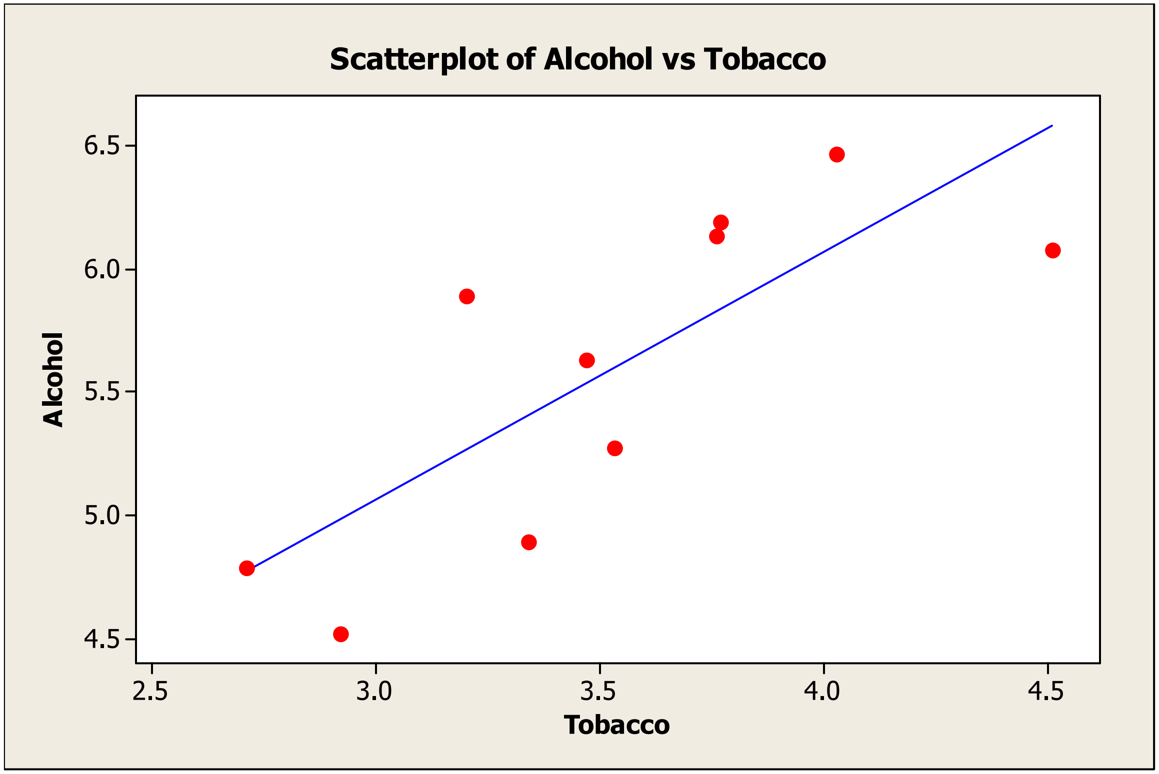
Why is it an outlier? It represents a different religion than other areas of Britain.



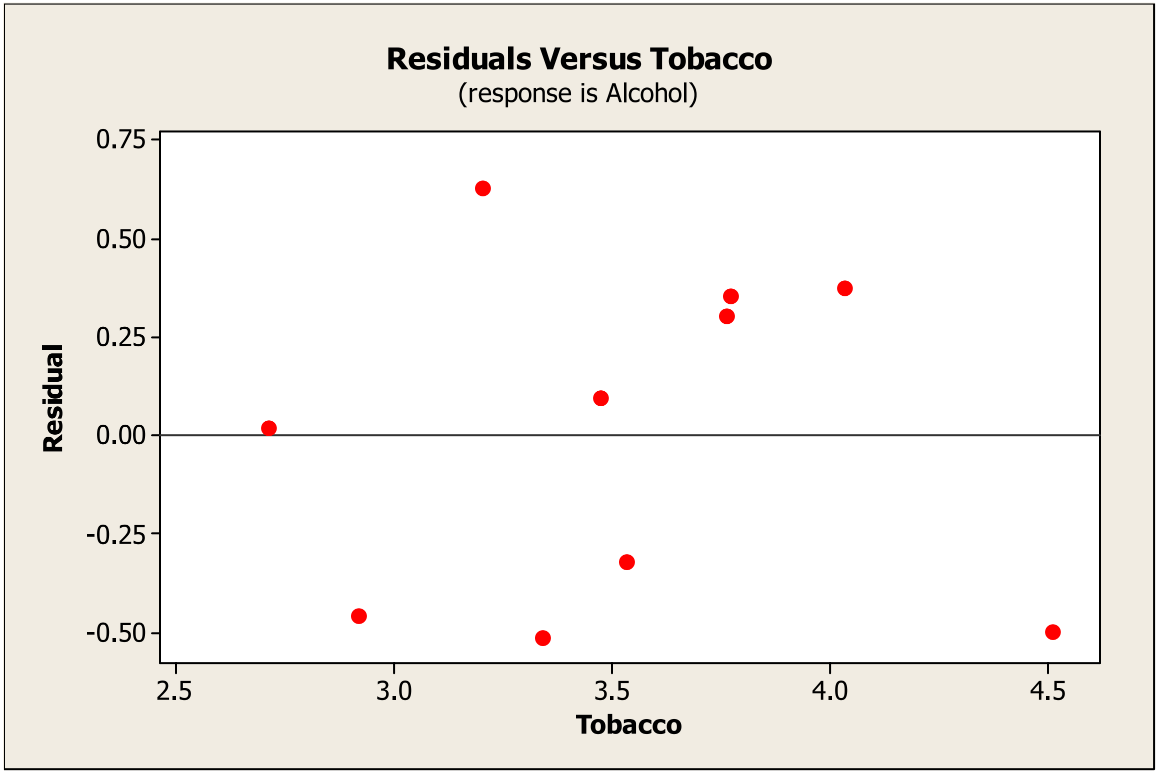
In the plot of residuals versus fitted values, it's even more obvious that the outlier is wreaking havoc.



The plot of residuals versus the X variable is very similar to residuals vs fitted values. Again the problem is obvious.



Here is a plot with Northern Ireland removed.



Here is a residual plot with Northern Ireland removed.

Notice how much the analysis changes when the outlier is removed:

With Outlier (Northern Ireland)

```
The regression equation is
Alcohol = 4.35 + 0.302 Tobacco

Predictor    Coef  SE Coef     T      P
Constant    4.351    1.607  2.71  0.024
Tobacco    0.3019   0.4388  0.69  0.509

S = 0.819630   R-Sq = 5.0%   R-Sq(adj) = 0.0%
```

Without Outlier

```
The regression equation is
Alcohol = 2.04 + 1.01 Tobacco

Predictor    Coef  SE Coef     T      P
Constant    2.041    1.001  2.04  0.076
Tobacco    1.0059   0.2813  3.58  0.007

S = 0.446020   R-Sq = 61.5%   R-Sq(adj) = 56.7%
```